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Research Paper

Tridiagonal Interval Matrix: Exploring New Perspectives and Application

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Abstract
Tridiagonal interval matrices are relevant in diverse applications, especially in dealing with parameter estimation, optimization and circuit analysis uncertainties. This research paper aims to improve the computational efficiency of obtaining the inverse of a general tridiagonal interval matrix. This matrix is pivotal in electric circuit analysis. We achieve this by employing interval arithmetic operations in the LU decomposition process, enabling effective handling of circuit parameter uncertainties. This approach generates an inverse interval matrix that addresses uncertainties in circuit analyses.

Keywords
Tridiagonal Interval Matrix, Generalized Interval Arithmetic, Interval LU Decomposition, Interval Determinants, Interval Matrix Inversion

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1. INTRODUCTION

Matrix inversion is fundamental operation in linear algebra that finds application in various fields, including electric circuit analysis. In practical applications, uncertainties and imprecisions often arise in the values of matrix entries, leading to the utilization of interval arithmetic and interval matrices. Interval matrices incorporate intervals as entries, representing the range of possible values for each element. In the context of electric circuit analysis, the inverse of a general tridiagonal interval matrix plays a crucial role in uncertainty quantification and numerical analysis, enabling the evaluation of circuit parameters under uncertain conditions. The inverse tridiagonal interval matrix with electric circuit applications has significant implications for uncertainty quantification, reliability analysis and robust circuit design. It allows evaluation of circuit parameters, such as voltages, currents and power, while considering uncertainties in component values and environmental conditions. Mallik (2001) presents a formula for the parts of the inverse of a general tridiagonal matrix. His method is based on linear difference equations with variable coefficients. El-Mikkawy and Karawia (2022) and El-Mikkawy (2004) devised a novel, non-restrictive method for determining the inverse of a general tridiagonal matrix. The foundation of his method is Doolittle LU factorization. Heydari et al. (2019) studied the inverse eigenvalues of symmetric acyclic matrices with generalized star graphs. Wei et al. (2019) present explicit formulas for the determinants and inverses of periodic tridiagonal Toeplitz matrices with perturbed corners. They used the Sherman-Morrison-Woodbury matrix decomposition. Encinas and Jiménez (2019) investigated the inverse of tridiagonal matrices, commonly referred to as Jacobi matrices. Furthermore, this approach is grounded in second-order linear difference equations. Wang and Jin (2019) investigated the properties of quasi-tridiagonal matrices with inverse eigenvalues. Spellacy et al. (2019) analyzed a complex block of tridiagonal matrices for partial inversion. Tan (2019) developed precise equations for finding the inverse of a specific category of tridiagonal matrices commonly found in interpolation problems and statistical models that use first-order autoregression to introduce covariance structure dependencies. Caratelli and Ricci (2021) presented method for inverting non-singular tridiagonal matrices using a classic functional analysis tool: Dunford-Taylor's integral, which adds operators to Cauchy's integral formula. Chu et al. (2022) proposed a modified inverse iteration method for computing symmetric tridiagonal eigenvectors. Hopkins and Kilic (2022) presented an algorithm for determining the inverse and the determinant of an extended, periodic, tridiagonal matrix. They are used in the fractional-backward approach. Parker et al. (2022) implemented a domain factorization algorithm similar to SPIKE and PDD. This approach relies primarily on Thomas algorithm as a direct inner solver. Talibi et al. (2022) developed an algorithm to calculate the inverse of a tridiagonal matrix based on the CL decomposition method. Almeida and Remigio (2023) presented sufficient conditions for the existence of the LU factorization of a Toeplitz symmetric tridiagonal matrix based on a modified Chebyshev polynomial and the parameters of

Crout's method. Bala et al. (2019) studied the inverse spectral problem of a tridiagonal matrix with almost symmetric spectra and spectral data. Fathi et al. (2020) investigated two distinct inverse eigenvalue problems related to a nonsymmetric tridiagonal matrix and explored the applications of these problems in graph theory and perturbation theory. Furtado et al. (2023) examined square matrices exhibiting the inverse diagonal property. Qi et al. (2019) provided two formulas for Chebyshev polynomials of the second kind. As well as integer powers, these formulas represent higher-order derivatives of the generated functions of the same Chebyshev polynomials. Kaucher (1980) introduced the concept of a monadic dual operator, which merges the duality principle, emphasizing that every element has an opposite, with the monadic principle, which states that any operations result should be a single element. Rohn (1993) suggested theoretical and practical approaches to computing the inverse interval matrix. Sengupta and Pal (2000) presented an easy-to-use and effective interval comparison technique. Ganesan and Veeramani (2005) developed an innovative collection of arithmetic operations for interval numbers with the overarching goal of mitigating disparities in a general context. Nirmala et al. (2011) devised an inventive approach for calculating the inverse of an interval matrix, which subsequently proved to be a valuable tool for addressing interval-linear problems. Hartman et al. (2021) introduced an algorithm for computing the spectral decomposition of an interval matrix. This algorithm provides an approximation to spectral decompositions for realizations of interval matrices. Thirupathi and Thamaraiselvan (2023) presented a set of computational algorithms known for their efficiency. These algorithms have been tailored to compute the determinant and inverse of general k-tridiagonal interval matrices. Furthermore, they include a symbolic algorithm for solving doubly bordered k-tridiagonal interval linear systems. These methods leverage LU decomposition and generalized interval arithmetic to tackle these mathematical problems. Following this inspiration and motivation, several authors, including, Susanti et al.; Sabri and Ahmed; and Yuan and Yang (2023), etc., have investigated uncertainty. This study investigates the inverse of a general tridiagonal interval matrix, focusing on its application in electric circuit analysis. To contribute to interval computation and uncertainty analysis, we examine the theoretical foundations, algorithmic approaches and practical considerations related to computing the inverse tridiagonal interval matrix. This paper is organized as follows: Section 2 gives an overview of generalized interval arithmetic. Section 3 presents a tridiagonal interval matrix and theorems. Section 4 provides algorithms for determining the inverse of a general tridiagonal interval matrix.

2. Preliminary Notes

The set of (proper and improper) generalized intervals we dente by $D = \{\mathbb{R} \cup \overline{\mathbb{R}} = [u_1, u_2] : u_1, u_2 \in \mathbb{R}\}$. The set of generalized intervals represented as D constitutes a group under both addition and multiplication operations, while simultaneously meeting the criteria of being devoid of zero and preserving inclusion monotonicity. The midpoint and width (or half-width) of an interval number $\tilde{u} = [u_1, u_2]$ are defined as $m(\tilde{u}) = \frac{u_1+u_2}{2}$ and $w(\tilde{u}) = \frac{u_2-u_1}{2}$. For $\tilde{u} = [u_1, u_2] \in D$, its dual is given by $\text{dual}(\tilde{u}) = \text{dual}[u_1, u_2] = [u_2, u_1]$. An interval's opposite $\tilde{u} = [u_1, u_2]$ is opp $\{[u_1, u_2]\} = [-u_1, -u_2]$ which is the additive inverse of $[u_1, u_2]$ and $\left[\frac{1}{u_1}, \frac{1}{u_2}\right]$ is the multiplicative inverse of $[u_1, u_2]$, provided $0 \notin [u_1, u_2]$. Thatis, $\tilde{u} + (-\text{dual}\,\tilde{u}) = \tilde{u} - \text{dual}(\tilde{u}) = [u_1, u_2] - \text{dual}([u_1, u_2] = [u_1, u_2] - [u_2, u_1] = [u_1 - u_1, u_2 - u_2] = [0, 0]$ and $\tilde{u} \times \left(\frac{1}{\text{dual}\,\tilde{u}}\right) = [u_1, u_2] \times \left(\frac{1}{\text{dual}([u_1, u_2])}\right) = [u_1, u_2] \times \frac{1}{[u_2, u_1]} = [u_1, u_2] \times \left[\frac{1}{u_1}, \frac{1}{u_2}\right] = [1, 1]$.

2.1 Interval Arithmetic

Ganesan and Veeramani (2005), introduced a novel approach to interval arithmetic. When expanding this arithmetic to the set of generalized interval numbers, D, we integrate the concept of dual. For $\tilde{u} = [u_1, u_2]$, $\tilde{v} = [v_1, v_2] \in D$ and for $* \in \{+, -, \cdot, \div\}$, we define $\tilde{u} * \tilde{v} = [m(\tilde{u}) * m(\tilde{v}) - j, m(\tilde{u}) * m(\tilde{v}) + j]$, where $j = \min\{(m(\tilde{u}) * m(\tilde{v})) - \beta, \gamma - (m(\tilde{u}) * m(\tilde{v}))\}$, β and γ define the endpoints of the interval $\tilde{u} \odot \tilde{v}$ in existing interval arithmetic. In particular,

1. Addition:
$\tilde{u}+\tilde{v} = [u_1, u_2]+[v_1, v_2] = [(m(\tilde{u})+m(\tilde{v}))-j, (m(\tilde{u})+m(\tilde{v}))+j]$,
where $j = \left\{\frac{(v_2+u_2)-(v_1+u_1)}{2}\right\}$.

2. Subtraction:
$\tilde{u}-\tilde{v} = [u_1, u_2]-[v_1, v_2] = [(m(\tilde{u})-m(\tilde{v}))-j, (m(\tilde{u})-m(\tilde{v}))+j]$,
where $j = \left\{\frac{(v_2+u_2)-(v_1+u_1)}{2}\right\}$.

Also if $\tilde{u} = \tilde{v}$, i.e. if $[u_1, u_2] = [v_1, v_2]$, then
$\tilde{u} - \tilde{v} = \tilde{u} - \text{dual}(\tilde{u}) = [u_1, u_2] - [u_2, u_1] = [u_1 - u_1, u_2 - u_2] = [0, 0]$.

3. Multiplication:
$\tilde{u}\cdot\tilde{v} = \tilde{u}\tilde{v} = [u_1, u_2]\cdot[v_1, v_2] = [(m(\tilde{u})m(\tilde{v}))-j, (m(\tilde{u})m(\tilde{v}))+j]$
where $j = \min\{(m(\tilde{u})m(\tilde{v})) - \beta, \gamma - (m(\tilde{u})m(\tilde{v}))\}$
$\beta = \min(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ and $m(u) = \left(\frac{u_1+u_2}{2}\right) \neq 0$

4. Division:
$1 \div \tilde{u} = \frac{1}{\tilde{u}} = \frac{1}{[u_1, u_2]} = \left[\frac{1}{m(\tilde{u})} - j, \frac{1}{m(\tilde{u})} + j\right]$,
where
$j = min 1/u_2((u_2 - u_1)/(u_1 + u_2)), 1/u_1((u_2 - u_1)/(u_1 + u_2))$
and $m(u) = \frac{u_1+u_2}{2} \neq 0$.
Also if $\tilde{u} = \tilde{v}$, i.e. if $[u_1, u_2] = [v_1, v_2]$, then
$\frac{\tilde{u}}{\tilde{v}} = \frac{\tilde{u}}{\tilde{u}} = \frac{\tilde{u}}{\text{dual}(\tilde{u})} = [u_1, u_2] \cdot \frac{1}{[u_2, u_1]} = [u_1, u_2] \cdot [\frac{1}{u_1}, \frac{1}{u_2}] = [1, 1]$.

5. Scalar multiplication:

$$\lambda\tilde{u} = \begin{cases} [\lambda u_1, \lambda u_2], & \text{jika } \lambda \geq 0 \\ [\lambda u_2, \lambda u_1], & \text{jika } \lambda < 0. \end{cases}$$

It is crucial to clarify that we use $\odot$ to represent the traditional interval arithmetic, while * is employed to indicate the modified interval arithmetic. In situations where there is no potential confusion, we may use the same notation for both. Furthermore, it is crucial to note that $\tilde{u} * \tilde{v} \subseteq \tilde{u} \odot \tilde{v}$, where $\odot \in \oplus, \ominus, \otimes, \oslash$, represents the traditional interval arithmetic.

Note 2.1. Without compromising generality, let us consider that for any interval number $\tilde{u} = [u_1, u_2]$ with $m(\tilde{u}) \neq 0$ and $0 \in \tilde{u}$, there exist $\tilde{v} = [m(\tilde{u}) - j, m(\tilde{u}) + j]$, where $0 < j < h$ and $h = \min\{|u_1|, |u_2|\}$, such that $\tilde{v} \approx \tilde{u}$ and $0 \notin \tilde{v}$. Hence, if $\frac{\tilde{a}}{\tilde{u}}$ with $m(\tilde{u}) \neq 0$ and $0 \in \tilde{u}$, then we replace $\frac{\tilde{a}}{\tilde{u}}$ by $\frac{\tilde{a}}{\tilde{v}}$ where $\tilde{v} \approx \tilde{u}$ and $0 \notin \tilde{v}$. Specifically, for convenience, one may choose j in a manner that

$$j = \begin{cases} \frac{m(\tilde{u})}{2}, & \text{if } m(\tilde{u}) > 0 \\ -\frac{m(\tilde{u})}{2}, & \text{if } m(\tilde{u}) < 0 \end{cases}$$

It is also crucial to emphasize that the distributive law for interval numbers can be established using this modified interval arithmetic, which in turn leads to numerous other significant discoveries.

2.2 Basics of Interval Matrices

Tridiagonal interval matrices are special kinds of matrices with a tridiagonal structure, with non-zero elements concentrated on the main diagonal and the diagonals immediately above and below it. The matrix also represents each element as an interval rather than as a real number. The lower and upper bounds of an interval are a set of real numbers, and they are often expressed as $[\underline{x}_i, \bar{x}_i]$, where $\underline{x}_i$ is the lower bound and $\bar{x}_i$ is the upper bound.

$$\tilde{A} = \begin{bmatrix} [\underline{d}_1, \bar{d}_1] & [\underline{a}_1, \bar{a}_1] & [0,0] & \cdots & \cdots & [0,0] \\ [\underline{b}_2, \bar{b}_2] & [\underline{d}_2, \bar{d}_2] & [\underline{a}_2, \bar{a}_2] & \vdots & \vdots & \vdots \\ [0,0] & [\underline{b}_3, \bar{b}_3] & [\underline{d}_3, \bar{d}_3] & [\underline{a}_3, \bar{a}_3] & [0,0] & \vdots \\ \vdots & [0,0] & \ddots & \ddots & \ddots & [0,0] \\ \vdots & \vdots & \vdots & \ddots & \ddots & [\underline{a}_{n-1}, \bar{a}_{n-1}] \\ [0,0] & \cdots & \cdots & [0,0] & [\underline{b}_n, \bar{b}_n] & [\underline{d}_n, \bar{d}_n] \end{bmatrix}$$

The midpoint of a tridiagonal interval matrix, denoted as $\tilde{A}$, is a matrix consisting of the midpoints of its interval elements defined as

$$m(\tilde{A}) = \begin{bmatrix} m(\tilde{d}_1) & m(\tilde{a}_1) & \tilde{0} & \cdots & \cdots & \tilde{0} \\ m(\tilde{b}_2) & m(\tilde{d}_2) & m(\tilde{a}_2) & \tilde{0} & \vdots & \vdots \\ \tilde{0} & m(\tilde{b}_3) & m(\tilde{d}_3) & \ddots & \ddots & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \ddots & \tilde{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & m(\tilde{a}_{n-1}) \\ \tilde{0} & \cdots & \cdots & \tilde{0} & m(\tilde{b}_n) & m(\tilde{d}_n) \end{bmatrix}$$

The width of a tridiagonal interval matrix, denoted as $\tilde{A}$, refers to a matrix that contains the widths of its interval elements defined as

$$w(\tilde{A}) = \begin{bmatrix} w(\tilde{d}_1) & w(\tilde{a}_1) & \tilde{0} & \cdots & \cdots & \tilde{0} \\ w(\tilde{b}_2) & w(\tilde{d}_2) & w(\tilde{a}_2) & \tilde{0} & \vdots & \vdots \\ \tilde{0} & w(\tilde{b}_3) & w(\tilde{d}_3) & \ddots & \ddots & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \ddots & \tilde{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & w(\tilde{b}_{n-1}) \\ \tilde{0} & \cdots & \cdots & \tilde{0} & w(\tilde{b}_n) & w(\tilde{d}_n) \end{bmatrix}$$

Let 0 be the null matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

and $\tilde{0}$ to represent the null interval matrix

$$\begin{bmatrix} \tilde{0} & \tilde{0} & \tilde{0} & \cdots & \cdots & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{0} & \cdots & \cdots & \vdots \\ \tilde{0} & \tilde{0} & \tilde{0} & \cdots & \cdots & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \tilde{0} \\ \tilde{0} & \cdots & \cdots & \cdots & \tilde{0} & \tilde{0} \end{bmatrix}$$

Also, 1 denotes the identity matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & 1 & \cdots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

and the identity interval matrix is indicated by $\tilde{1}$

$$\begin{bmatrix} \tilde{1} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \tilde{1} & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & \tilde{1} & \cdots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \tilde{1} \end{bmatrix}.$$

If the midpoint of a tridiagonal interval matrix $\tilde{A}$ is equal to the midpoint of a tridiagonal interval matrix $\tilde{B}$, then these matrices, $\tilde{A}$ and $\tilde{B}$, are considered equivalent and are represented as $\tilde{A} \approx \tilde{B}$. Specifically, when both the midpoint and the width of interval matrices $\tilde{A}$ and $\tilde{B}$ are equal, it can be expressed as $\tilde{A} = \tilde{B}$. If the midpoint of an interval matrix $\tilde{A}$ is zero, it is referred to as a zero-interval matrix. If both the midpoint and width of $\tilde{A}$ are zero, it can be denoted as $\tilde{A} = \tilde{0}$. When the midpoint of $\tilde{A}$ is zero, but the width of $\tilde{A}$ is a non-zero interval matrix, $\tilde{A}$ is considered a non-zero interval matrix. $\tilde{A}$ is classified as an identity interval matrix when its midpoint is equal to I. If the midpoint is I and the width is 0, it can be represented as $\tilde{A} = \tilde{I}$. If the midpoint is I and the width is not zero, it can be denoted as $\tilde{A} \approx \tilde{I}$.

2.3 Arithmetic Operations on Interval Matrices

If $\tilde{A}, \tilde{B} \in D^{m\times n}$, $\tilde{\mathbf{x}} \in D^n$ and $\tilde{\alpha} \in D$, then

1. $\tilde{\alpha}\tilde{A} \approx (\alpha\tilde{a}_{ij})$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$
2. $\tilde{A} + \tilde{B} \approx (\tilde{a}_{ij} + \tilde{b}_{ij})$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$
3. $\tilde{A} - \tilde{B} \approx \begin{cases} (\tilde{a}_{ij} - \tilde{b}_{ij})_{(1\le i\le m, 1\le j\le n)}, & \text{if } \tilde{A} \not\approx \tilde{B} \\ \tilde{A} - \text{dual}(\tilde{A}) \approx \tilde{O} = \tilde{O}, & \text{if } \tilde{A} \approx \tilde{B} \end{cases}$
4. $\tilde{A}\tilde{B} \approx \left(\sum_{k=1}^{n} \tilde{a}_{ik}\tilde{b}_{kj}\right)$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$
5. $\tilde{A}\tilde{\mathbf{x}} \approx \left(\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{\mathbf{x}}\right)$ for $i = 1, 2, \dots, n$.

3. LU Decomposition of Tridiagonal Interval Matrices

The interval Crout $\tilde{L}\tilde{U}$ factorization of the tridiagonal interval matrix $\tilde{A}$ is gives

$$\tilde{A} \approx \tilde{L}\tilde{U} \tag{1}$$

where

$$\tilde{A} = \begin{bmatrix} [\underline{d}_1, \bar{d}_1] & [\underline{a}_1, \bar{a}_1] & [0,0] & \cdots & \cdots & [0,0] \\ [\underline{b}_2, \bar{b}_2] & [\underline{d}_2, \bar{d}_2] & [\underline{a}_2, \bar{a}_2] & \vdots & \vdots & \vdots \\ [0,0] & [\underline{b}_3, \bar{b}_3] & [\underline{d}_3, \bar{d}_3] & [\underline{a}_3, \bar{a}_3] & [0,0] & \vdots \\ \vdots & [0,0] & \ddots & \ddots & \ddots & [0,0] \\ \vdots & \vdots & \vdots & \ddots & \ddots & [\underline{a}_{(n-1)}, \bar{a}_{(n-1)}] \\ [0,0] & \cdots & \cdots & [0,0] & [\underline{b}_n, \bar{b}_n] & [\underline{d}_n, \bar{d}_n] \end{bmatrix}$$

$$\tilde{L} = \begin{bmatrix} [\underline{l}_{11}, \bar{l}_{11}] & [0,0] & [0,0] & \cdots & \cdots & [0,0] \\ [\underline{l}_{21}, \bar{l}_{21}] & [\underline{l}_{22}, \bar{l}_{22}] & [0,0] & \cdots & \cdots & \vdots \\ [0,0] & [\underline{l}_{32}, \bar{l}_{32}] & [\underline{l}_{33}, \bar{l}_{33}] & \cdots & \cdots & \vdots \\ \vdots & [0,0] & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & [0,0] \\ [0,0] & \cdots & \cdots & \cdots & [\underline{l}_{n(n-1)}, \bar{l}_{n(n-1)}] & [\underline{l}_{nn}, \bar{l}_{nn}] \end{bmatrix}$$

$$\tilde{U} = \begin{bmatrix} [1,1] & [\underline{u}_{12}, \bar{u}_{12}] & [0,0] & \cdots & \cdots & [0,0] \\ [0,0] & [1,1] & [\underline{u}_{23}, \bar{u}_{23}] & \cdots & \cdots & \vdots \\ [0,0] & [0,0] & [1,1] & \cdots & \cdots & \vdots \\ \vdots & [0,0] & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & [\underline{u}_{(n-1)n}, \bar{u}_{(n-1)n}] \\ [0,0] & \cdots & \cdots & \cdots & [0,0] & [1,1] \end{bmatrix}$$

$$\tilde{L}\tilde{U} \approx \begin{bmatrix} \tilde{l}_{11} & \tilde{l}_{11}\tilde{u}_{12} & \tilde{0} & \cdots & \cdots & \tilde{0} \\ \tilde{l}_{12} & \tilde{l}_{21}\tilde{u}_{12} + \tilde{l}_{22} & \tilde{l}_{22}\tilde{u}_{23} & \cdots & \cdots & \vdots \\ \tilde{0} & \tilde{l}_{32} & \tilde{l}_{32}\tilde{u}_{23} + \tilde{l}_{33} & \tilde{l}_{33}\tilde{u}_{34} & \cdots & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \tilde{l}_{(n-1)(n-1)}\tilde{u}_{(n-1)n} \\ \tilde{0} & \cdots & \cdots & \cdots & \tilde{l}_{n(n-1)} & \tilde{l}_{n(n-1)}\tilde{u}_{(n-1)n} + \tilde{l}_{nn} \end{bmatrix}$$

In order to find the entries of interval matrices $\tilde{L}$ and $\tilde{U}$ dual subtraction and dual division are used as follows:

$$\begin{aligned} \tilde{d}_1 &= \tilde{l}_{11} \\ \tilde{a}_1 &= \tilde{l}_{11}\tilde{u}_{12} \Rightarrow \tilde{u}_{12} = \frac{\tilde{a}_1}{\tilde{l}_{11}} \\ \tilde{b}_2 &= \tilde{l}_{21} \\ \tilde{d}_2 &= \tilde{l}_{21}\tilde{u}_{12} + \tilde{l}_{22} \Rightarrow \tilde{d}_2 = \tilde{b}_2\frac{\tilde{a}_1}{\tilde{l}_{11}} + \tilde{l}_{22} \\ &\Rightarrow \tilde{d}_2 - \tilde{b}_2\frac{\tilde{a}_1}{\tilde{l}_{11}} = \tilde{l}_{22} \end{aligned}$$

$$\begin{aligned} \tilde{a}_2 &= \tilde{l}_{22}\tilde{u}_{23} \Rightarrow \tilde{u}_{23} = \frac{\tilde{a}_2}{\tilde{l}_{22}} \\ \tilde{b}_3 &= \tilde{l}_{32} \\ \tilde{d}_3 &= \tilde{l}_{32}\tilde{u}_{23} + \tilde{l}_{33} \Rightarrow \tilde{d}_3 = \tilde{b}_3\frac{\tilde{a}_2}{\tilde{l}_{22}} + \tilde{l}_{33} \\ &\Rightarrow \tilde{d}_3 - \tilde{b}_3\frac{\tilde{a}_2}{\tilde{l}_{22}} = \tilde{l}_{33} \end{aligned}$$

$$\begin{aligned} \tilde{a}_3 &= \tilde{l}_{33}\tilde{u}_{34} \Rightarrow \tilde{u}_{34} = \frac{\tilde{a}_3}{\tilde{l}_{33}} \\ \tilde{b}_4 &= \tilde{l}_{42} \\ \tilde{d}_4 &= \tilde{l}_{42}\tilde{u}_{34} + \tilde{l}_{44} \Rightarrow \tilde{d}_4 = \tilde{b}_4\frac{\tilde{a}_3}{\tilde{l}_{33}} + \tilde{l}_{44} \\ &\Rightarrow \tilde{d}_4 - \tilde{b}_4\frac{\tilde{a}_3}{\tilde{l}_{33}} = \tilde{l}_{44} \end{aligned}$$

$$\begin{aligned} \tilde{a}_4 &= \tilde{l}_{44}\tilde{u}_{45} \Rightarrow \tilde{u}_{45} = \frac{\tilde{a}_4}{\tilde{l}_{44}} \\ \tilde{b}_5 &= \tilde{l}_{54} \\ \tilde{d}_5 &= \tilde{l}_{54}\tilde{u}_{45} + \tilde{l}_{55} \Rightarrow \tilde{d}_5 = \tilde{b}_5\frac{\tilde{a}_4}{\tilde{l}_{44}} + \tilde{l}_{55} \\ &\Rightarrow \tilde{d}_5 - \tilde{b}_5\frac{\tilde{a}_4}{\tilde{l}_{44}} = \tilde{l}_{55} \end{aligned}$$

$$\begin{aligned}
&\tilde{a}_5 = \tilde{l}_{44}\tilde{u}_{56} \Rightarrow \tilde{u}_{56} = \frac{\tilde{a}_5}{\tilde{l}_{55}}\\
&\tilde{b}_6 = \tilde{l}_{65}\\
&\tilde{d}_5 = \tilde{l}_{54}\tilde{u}_{45} + \tilde{l}_{55} \Rightarrow \tilde{d}_5 = \tilde{b}_5\frac{\tilde{a}_4}{\tilde{l}_{44}} + \tilde{l}_{55}\\
&\quad \Rightarrow \tilde{d}_6 - \tilde{b}_6\frac{\tilde{a}_5}{\tilde{l}_{55}} = \tilde{l}_{66}
\end{aligned}$$

$$\begin{aligned}
&\tilde{a}_6 = \tilde{l}_{66}\tilde{u}_{67} \Rightarrow \tilde{u}_{67} = \frac{\tilde{a}_6}{\tilde{l}_{66}}\\
&\tilde{b}_7 = \tilde{l}_{76}\\
&\tilde{d}_7 = \tilde{l}_{76}\tilde{u}_{67} + \tilde{l}_{77} \Rightarrow \tilde{d}_7 = \tilde{b}_7\frac{\tilde{a}_6}{\tilde{l}_{66}} + \tilde{l}_{77}\\
&\quad \Rightarrow \tilde{d}_7 - \tilde{b}_7\frac{\tilde{a}_6}{\tilde{l}_{66}} = \tilde{l}_{77}
\end{aligned}$$

Let $k_1 = \tilde{l}_{11}, \quad k_2 = \tilde{l}_{22}, \quad k_3 = \tilde{l}_{33}, \quad \ldots, \quad k_n = \tilde{l}_{nn}$. In general,

$$\tilde{k}_i = \begin{cases} \tilde{d}_1 & \text{if } i = 1, \\ \tilde{d}_i - \frac{\tilde{b}_i}{\tilde{k}_{i-1}}\tilde{a}_{i-1} & \text{if } i = 2, 3, \ldots, n. \end{cases} \tag{2}$$

Theorem 3.1. If, for each For $i = 1, 2, 3, \ldots, n, \quad m(\tilde{k}_i) \neq \tilde{0}$, then the tridiagonal interval matrix $\tilde{A}$ given by Equation (1) has at least two LU factorizations, say

$$\tilde{A} \approx \tilde{L}_1\tilde{U}_1 \approx \tilde{L}_2\tilde{U}_2$$

where

$$\tilde{L}_1 = \begin{bmatrix}
[1,1] & [0,0] & [0,0] & \cdots & \cdots & [0,0]\\
\frac{[\underline{b}_2,\bar{b}_2]}{[\underline{k}_1,\bar{k}_1]} & [1,1] & [0,0] & \ddots & \ddots & \vdots\\
[0,0] & \frac{[\underline{b}_3,\bar{b}_3]}{[\underline{k}_2,\bar{k}_2]} & [1,1] & \ddots & [0,0] & \vdots\\
\vdots & [0,0] & \ddots & \ddots & \ddots & [0,0]\\
\vdots & \vdots & \ddots & \ddots & \ddots & [0,0]\\
[0,0] & \cdots & \cdots & [0,0] & \frac{[\underline{b}_n,\bar{b}_n]}{[\underline{k}_{n-1},\bar{k}_{n-1}]} & [1,1]
\end{bmatrix}$$

$$\tilde{U}_1 = \begin{bmatrix}
[\underline{k}_1,\bar{k}_1] & [\underline{a}_1,\bar{a}_1] & [0,0] & \cdots & \cdots & [0,0]\\
[0,0] & [\underline{k}_2,\bar{k}_2] & [\underline{a}_2,\bar{a}_2] & \ddots & \ddots & \vdots\\
[0,0] & [0,0] & [\underline{k}_3,\bar{k}_3] & \ddots & [0,0] & \vdots\\
\vdots & [0,0] & \ddots & \ddots & \ddots & [0,0]\\
[0,0] & [0,0] & \cdots & \cdots & [0,0] & [\underline{a}_{n-1},\bar{a}_{n-1}]\\
[0,0] & \cdots & \cdots & [0,0] & [0,0] & [\underline{k}_n,\bar{k}_n]
\end{bmatrix} \tag{3}$$

$$\tilde{L}_2 = \begin{bmatrix}
[\underline{k}_1,\bar{k}_1] & [0,0] & [0,0] & \cdots & \cdots & [0,0]\\
[\underline{b}_2,\bar{b}_2] & [\underline{k}_2,\bar{k}_2] & [0,0] & \ddots & \ddots & \vdots\\
[0,0] & [\underline{b}_3,\bar{b}_3] & \ddots & \ddots & [0,0] & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & [0,0]\\
\vdots & \vdots & \ddots & \ddots & \ddots & [0,0]\\
[0,0] & \cdots & \cdots & [0,0] & [\underline{b}_n,\bar{b}_n] & [\underline{k}_n,\bar{k}_n]
\end{bmatrix}$$

$$\tilde{U}_2 = \begin{bmatrix}
[1,1] & \frac{[\underline{a}_1,\bar{a}_1]}{[\underline{k}_1,\bar{k}_1]} & [0,0] & \cdots & \cdots & [0,0]\\
[0,0] & [1,1] & \frac{[\underline{a}_2,\bar{a}_2]}{[\underline{k}_2,\bar{k}_2]} & \ddots & \ddots & \vdots\\
[0,0] & [0,0] & [1,1] & \ddots & [0,0] & \vdots\\
\vdots & [0,0] & \ddots & \ddots & \ddots & [0,0]\\
\vdots & \vdots & \ddots & \ddots & \ddots & \frac{[\underline{a}_{n-1},\bar{a}_{n-1}]}{[\underline{k}_{n-1},\bar{k}_{n-1}]}\\
[0,0] & \cdots & \cdots & [0,0] & [0,0] & [1,1]
\end{bmatrix} \tag{4}$$

Note 3.1. The two factorizations $\tilde{L}_1\tilde{U}_1$ and $\tilde{L}_2\tilde{U}_2$, given by Equation (3) and (4), are called interval Doolittle and Crouts factorizations.

Corollary 3.1. Let $\tilde{A}$ be a tridiagonal interval matrix given in (1). If, for each $i = 1, 2, 3, \ldots, n-1, \quad m(\tilde{k}_i) \neq \tilde{0}$, then we have

$$\det(\tilde{A}) \approx [\underline{k}_1, \bar{k}_1] \times [\underline{k}_2, \bar{k}_2] \times \ldots \times [\underline{k}_n, \bar{k}_n]$$

Proof. If $\tilde{A}$ can be factored into lower triangular interval matrix $\tilde{L}$ and upper triangular interval matrix $\tilde{U}$ such that

$$\tilde{A} \approx \tilde{L}\tilde{U}$$

Now

$$\det(\tilde{A}) \approx \det(\tilde{L}\tilde{U}) \approx \det(\tilde{L})\det(\tilde{U})$$

Where

$$\tilde{L}_1 = \begin{bmatrix}
[1,1] & [0,0] & [0,0] & \cdots & \cdots & [0,0]\\
\frac{[\underline{b}_2,\bar{b}_2]}{[\underline{k}_1,\bar{k}_1]} & [1,1] & [0,0] & \ddots & \ddots & \vdots\\
[0,0] & \frac{[\underline{b}_3,\bar{b}_3]}{[\underline{k}_2,\bar{k}_2]} & [1,1] & \ddots & [0,0] & \vdots\\
\vdots & [0,0] & \ddots & \ddots & \ddots & [0,0]\\
\vdots & \vdots & \ddots & \ddots & \ddots & [0,0]\\
[0,0] & \cdots & \cdots & [0,0] & \frac{[\underline{b}_n,\bar{b}_n]}{[\underline{k}_{n-1},\bar{k}_{n-1}]} & [1,1]
\end{bmatrix}$$

$\tilde{L}$ is the lower triangular interval with diagonal entries [1,1], …, [1,1].

$$\tilde{U} \approx \begin{bmatrix}
[\underline{k}_1,\overline{k}_1] & [\underline{a}_1,\overline{a}_1] & [0,0] & \cdots & \cdots & [0,0]\\
[0,0] & [\underline{k}_2,\overline{k}_2] & [\underline{a}_2,\overline{a}_2] & \ddots & \ddots & \vdots\\
[0,0] & [0,0] & [\underline{k}_3,\overline{k}_3] & \ddots & [0,0] & \vdots\\
\vdots & [0,0] & \ddots & \ddots & \ddots & [0,0]\\
\vdots & \vdots & \ddots & \ddots & \ddots & [\underline{a}_{n-1},\overline{a}_{n-1}]\\
[0,0] & \ldots & \cdots & [0,0] & [0,0] & [\underline{k}_n,\overline{k}_n]
\end{bmatrix}$$

Using mathematical induction on the size of $\widetilde{U}$, clearly it is true if $\widetilde{U}$ is a 1×1 interval matrix. Suppose that it is true when $\widetilde{U}$ is an $(n-1) \times (n-1)$ interval matrix where $n \geq 2$. We show that it is true for when $\widetilde{U}$ is an $n \times n$ interval matrix. Note that expanding along the first column gives us that $\det(\widetilde{U}) \approx [\underline{k}_1, \overline{k}_1] \det(\widetilde{U}_{11})$, where $\widetilde{U}_{11}$ is the interval matrix obtained by deleting the first row and first column of $\widetilde{U}$. Now note that $\widetilde{U}_{11}$ is an $(n-1) \times (n-1)$ upper triangular interval matrix with diagonal entries $[\underline{k}_2, \overline{k}_2]$, $[\underline{k}_3, \overline{k}_3]$, ..., $[\underline{k}_n, \overline{k}_n]$. By the induction hypothesis,

$$\det(\widetilde{U}) \approx [\underline{k}_1, \overline{k}_1] \times [\underline{k}_2, \overline{k}_2] \times [\underline{k}_3, \overline{k}_3] \times \ldots \times [\underline{k}_n, \overline{k}_n].$$

$$\det(\widetilde{A}) \approx \det(\widetilde{L}) \det(\widetilde{U})$$

$$\approx [1, 1] \times [\underline{k}_1, \overline{k}_1] \times [\underline{k}_2, \overline{k}_2] \times [\underline{k}_3, \overline{k}_3] \times \ldots \times [\underline{k}_n, \overline{k}_n].$$

$$\det(\widetilde{A}) \approx [\underline{k}_1, \overline{k}_1] \times [\underline{k}_2, \overline{k}_2] \times [\underline{k}_3, \overline{k}_3] \times \ldots \times [\underline{k}_n, \overline{k}_n].$$

Theorem 3.2. Any real symmetric tridiagonal interval matrix

$$\begin{bmatrix} [\underline{d}_1, \overline{d}_1] & [\underline{a}_1, \overline{a}_1] & [0,0] & \cdots & \cdots & [0,0] \\ [\underline{a}_1, \overline{a}_1] & [\underline{d}_2, \overline{d}_2] & [\underline{a}_2, \overline{a}_2] & \cdots & \ddots & \vdots \\ [0,0] & [\underline{a}_2, \overline{a}_2] & [\underline{d}_3, \overline{d}_3] & \cdots & \vdots & \vdots \\ \vdots & [0,0] & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & [\underline{a}_{n-1}, \overline{a}_{n-1}] & \vdots \\ [0,0] & \cdots & \cdots & \cdots & [\underline{a}_{n-1}, \overline{a}_{n-1}] & [\underline{d}_n, \overline{d}_n] \end{bmatrix} \tag{5}$$

is positive definite if and only if $m(\widetilde{k}_i) > 0$ for each $i = 1, 2, 3, \ldots, n$.

Note 3.2. For the tridiagonal interval matrix defined in Equation (5), if $m(\widetilde{k}_i) > 0$ for each $i = 1, 2, 3, \ldots, n$, it is always possible to perform an $\widetilde{L}\widetilde{U}$ factorization in the form of $\widetilde{L}\widetilde{L}^t$. This specific factorization is known as the interval Cholesky factorization.

Theorem 3.3. Let $\widetilde{T}_k$ be a tridiagonal interval matrix of the form

$$\widetilde{T}_k = \begin{bmatrix} [\underline{d}_1, \overline{d}_1] & [\underline{a}_1, \overline{a}_1] & \tilde{0} & \cdots & \cdots & 0 \\ [\underline{b}_2, \overline{b}_2] & [\underline{d}_2, \overline{d}_2] & [\underline{a}_2, \overline{a}_2] & \cdots & \cdots & \vdots \\ \tilde{0} & [\underline{b}_3, \overline{b}_3] & [\underline{d}_3, \overline{d}_3] & \cdots & \cdots & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & [\underline{a}_{k-1}, \overline{a}_{k-1}] \\ \tilde{0} & \cdots & \cdots & \cdots & [\underline{b}_k, \overline{b}_k] & [\underline{d}_k, \overline{d}_k] \end{bmatrix}$$

where $k = 1, 2, 3, \ldots, n$. Then the tridiagonal interval matrix $\widetilde{T}_k$ satisfy three term recurrence relation:

$$\det(\widetilde{T}_1) = [\underline{d}_1, \overline{d}_1]$$

$$\det(\widetilde{T}_2) = [\underline{d}_1, \overline{d}_1][\underline{d}_2, \overline{d}_2] - [\underline{a}_1, \overline{a}_1][\underline{b}_2, \overline{b}_2]$$

$$\det(\widetilde{T}_k) = [\underline{d}_k, \overline{d}_k]\det(\widetilde{T}_{k-1}) - [\underline{a}_{k-1}, \overline{a}_{k-1}][\underline{b}_k, \overline{b}_k]\det(\widetilde{T}_{k-2}), \quad k \geq 3.$$

Proof. Using mathematical induction, it is simple to confirm that the theorem is true for $k = 1$ and 2.

Now

$$\det(\widetilde{T}_k) = \begin{vmatrix} [\underline{d}_1, \overline{d}_1] & [\underline{a}_1, \overline{a}_1] & \tilde{0} & \cdots & \cdots & \tilde{0} \\ [\underline{b}_2, \overline{b}_2] & [\underline{d}_2, \overline{d}_2] & [\underline{a}_2, \overline{a}_2] & \cdots & \cdots & \vdots \\ \tilde{0} & [\underline{b}_3, \overline{b}_3] & [\underline{d}_3, \overline{d}_3] & \cdots & \cdots & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & [\underline{a}_{k-1}, \overline{a}_{k-1}] \\ \tilde{0} & \cdots & \cdots & \cdots & [\underline{b}_k, \overline{b}_k] & [\underline{d}_k, \overline{d}_k] \end{vmatrix}$$

Using cofactor expansion on the last column and row,

$$\det(\widetilde{T}_k) = [\underline{d}_k, \overline{d}_k]\det(\widetilde{T}_{k-1})[\underline{a}_{k-1}, \overline{a}_{k-1}]\det \begin{vmatrix} [\underline{d}_1, \overline{d}_1] & [\underline{a}_1, \overline{a}_1] & \tilde{0} & \cdots & \tilde{0} \\ [\underline{b}_2, \overline{b}_2] & [\underline{d}_2, \overline{d}_2] & [\underline{a}_2, \overline{a}_2] & \cdots & \vdots \\ \tilde{0} & [\underline{b}_3, \overline{b}_3] & [\underline{d}_3, \overline{d}_3] & \cdots & \vdots \\ \vdots & \tilde{0} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & [\underline{a}_{k-1}, \overline{a}_{k-1}] \\ \tilde{0} & \cdots & \cdots & \cdots & [\underline{d}_k, \overline{d}_k] \end{vmatrix}$$

$$|\det(\widetilde{T}_k)| = [\underline{d}_k, \overline{d}_k]\det(\widetilde{T}_{k-1}) - [\underline{a}_{k-1}, \overline{a}_{k-1}][\underline{b}_k, \overline{b}_k]\det(\widetilde{T}_{k-2}) \tag{6}$$

where the initial values for $(\widetilde{T}_k)$ are $(\widetilde{T}_0)$ =[1,1] and $(\widetilde{T}_{-1})$ =[0,0].

Theorem 3.4. If, for each i=1,2,3,...,n-1, $m(\widetilde{k}_i) > \tilde{0}$, the interval of the three-term recurrence given in Equation (6) reduces to the interval of a two-term recurrence.

$$|(\widetilde{T}_i)| = [\underline{k}_i, \overline{k}_i]|(\widetilde{T}_{i-1})| \quad \text{for each } i = 1, 2, 3, \ldots, n.$$

Proof. By using (2) and (6)

$$\tilde{k}_i = \begin{cases} \tilde{d}_1, & \text{if } i = 1, \\ \tilde{d}_i - \frac{b_i}{\tilde{k}_{i-1}} \tilde{a}_{i-1}, & \text{if } i = 2, 3, \ldots, n. \end{cases}$$

$$[\underline{k}_i, \overline{k}_i] = [\underline{d}_i, \overline{d}_i] - \frac{[\underline{b}_i, \overline{b}_i]}{[\underline{k}_{i-1}, \overline{k}_{i-1}]}[\underline{a}_{i-1}, \overline{a}_{i-1}]$$

$$= \frac{[\underline{d}_i, \overline{d}_i][\underline{k}_{i-1}, \overline{k}_{i-1}] - [\underline{b}_i, \overline{b}_i][\underline{a}_{i-1}, \overline{a}_{i-1}]}{[\underline{k}_{i-1}, \overline{k}_{i-1}]}$$

$$[\underline{k}_{(i-1)},\overline{k}_{(i-1)}]=[\underline{d}_i,\overline{d}_i][\underline{k}_{(i-1)},\overline{k}_{(i-1)}]-[\underline{b}_i,\overline{b}_i][\underline{a}_{(i-1)},\overline{a}_{(i-1)}]$$

$$[\underline{a}_{(i-1)},\overline{a}_{(i-1)}]=[\underline{d}_i,\overline{d}_i][\underline{k}_{(i-1)},\overline{k}_{(i-1)}]-[\underline{k}_i,\overline{k}_i][\underline{k}_{(i-1)},\overline{k}_{(i-1)}]. \quad (7)$$

Utilising (7) in (6)

$$|(\widetilde{T}_i)|=[\underline{d}_i,\overline{d}_i]|(\widetilde{T}_{i-1})|-\Big([\underline{d}_i,\overline{d}_i][\underline{k}_{i-1},\overline{k}_{i-1}]-[\underline{k}_i,\overline{k}_i][\underline{k}_{i-1},\overline{k}_{i-1}]\Big)|(\widetilde{T}_{i-2})|$$

$$|\widetilde{T}_i|=\Big(|\widetilde{T}_{i-1}|-[\underline{k}_{i-1},\overline{k}_{i-1}]|\widetilde{T}_{i-2}|\Big)[\underline{d}_i,\overline{d}_i]+[\underline{k}_i,\overline{k}_i][\underline{k}_{i-1},\overline{k}_{i-1}]|\widetilde{T}_{i-2}|, \quad (8)$$

for each $i = 1, 2, \cdots, n$.
For $i = 2$, in (8) gives:

By using dual subtraction, we get

$$|\widetilde{T}_2| = [\underline{k}_2, \overline{k}_2]|\widetilde{T}_1|.$$

For i=3, in (8) gives:

$$|\widetilde{T}_3| = [\underline{k}_3, \overline{k}_3]|\widetilde{T}_2|,$$

and so on. For i=n, in (8) gives:

$$|\widetilde{T}_n| = [\underline{k}_n, \overline{k}_n|\widetilde{T}_{n-1}|.$$

4. An algorithm for the Inverse of a General Tridiagonal Interval Matrix

This section proposes generalizing a novel computational algorithm for inverting any nonsingular tridiagonal interval matrix.

4.1 Algorithm for Inverse Tridiagonal Interval Matrix

The inverse of the tridiagonal interval matrix $\widetilde{W} = \widetilde{A}^{-1} of = \widetilde{A}$, where $\widetilde{W} = [\underline{w}_{ij}, \overline{w}_{ij}]_{1\le i,j\le n}$.

Step 1.

Compute:

$$\tilde{k}_i = \begin{cases} \tilde{d}_1, & \text{if } i = 1, \\ \tilde{d}_i - \frac{\tilde{b}_i}{\tilde{k}_{i-1}}\tilde{a}_{i-1}, & \text{if } i = 2, 3, \ldots, n \end{cases}$$

If $m(\tilde{k}_i)\neq 0$ for each i=1,2,3,$\cdots$,n, then the tridiagonal interval matrix $\tilde{A}$ is nonsingular.

Step 2.

If i=1,2,$\cdots$,n, compute and simplify the intervals

$$[\underline{v}_{ii}, \overline{v}_{ii}] = [1, 1]$$
$$[\underline{u}_{ii}, \overline{u}_{ii}] = \frac{[1,1]}{[\underline{k}_i, \overline{k}_i]}$$

Step 3.

If $j = 2, 3, \cdots, n$, for $i = j-1, \cdots, 1$ compute the intervals

$$[\underline{v}_{ij}, \overline{v}_{ij}] = [0, 0]$$
$$[\underline{u}_{ij}, \overline{u}_{ij}] = -\frac{[\underline{a}_i, \overline{a}_i]}{[\underline{k}_i, \overline{k}_i]}[\underline{u}_{(i+1,j)}, \overline{u}_{(i+1,j)}]$$

Step 4.

If $i = 2, 3, \cdots, n$, for$j = 1, 2, \cdots, i-1$ compute and simplify the intervals

$$[\underline{u}_{ij}, \overline{u}_{ij}] = [0, 0]$$
$$[\underline{v}_{ij}, \overline{v}_{ij}] = -\frac{[\underline{b}_i, \overline{b}_i]}{[\underline{k}_{(i-1)}, \overline{k}_{(i-1)}]}[\underline{v}_{(i-1,j)}, \overline{v}_{(i-1,j)}]$$

Step 5.

If $i = 1, 2, \cdots, n$, and$j = 1, 2, \cdots, n$, calculate and simplify the interval numbers

$$[\underline{w}_{ij}, \overline{w}_{ij}] = \sum_{k=\max(i,j)}^{n}[\underline{u}_{ik}, \overline{u}_{ik}][\underline{v}_{kj}, \overline{v}_{kj}]$$

Example 4.1. Find the inverse of the tridiagonal interval matrix

$$\widetilde{B} = \begin{bmatrix} [2.251, 2.273] & [-2.273, -2.251] & [0, 0] & [0, 0] \\ [-2.273, -2.251] & [4.502, 4.546] & [-2.273, -2.251] & [0, 0] \\ [0, 0] & [-2.273, -2.251] & [6.639, 6.684] & [-4.411, -4.388] \\ [0, 0] & [0, 0] & [-4.411, -4.388] & [8.776, 8.822] \end{bmatrix}.$$

Applying the tridiagonal interval matrix inverse algorithm yields:
By using **step 1**, we get:

$$\widetilde{k_1} = [2.251, 2.273], \widetilde{k_2} = [2.209, 2.318],$$
$$\widetilde{k_3} = [4.305, 4.496], \widetilde{k_4} = [4.264, 4.548].$$
$$\det(\widetilde{B}) = [91.267, 107.272] \not\approx 0.$$

Then the given tridiagonal interval matrix $\tilde{B}$ is non-singular. By using **Steps 2-5**, we get

$$\widetilde{B}^{-1} = \begin{bmatrix} [1.263, 1.411] & [0.831, 0.959] & [0.415, 0.492] & [0.206, 0.247] \\ [0.831, 0.959] & [0.839, 0.951] & [0.419, 0.488] & [0.208, 0.245] \\ [0.415, 0.491] & [0.419, 0.487] & [0.430, 0.476] & [0.214, 0.239] \\ [0.206, 0.247] & [0.208, 0.245] & [0.214, 0.239] & [0.220, 0.234] \end{bmatrix}$$

4.2 Application on Electric Circuit

A network equation must be constructed to determine the electrical current flow in a circuit. In real life, not all decision parameters may be known precisely so that they may be treated as suitable intervals.

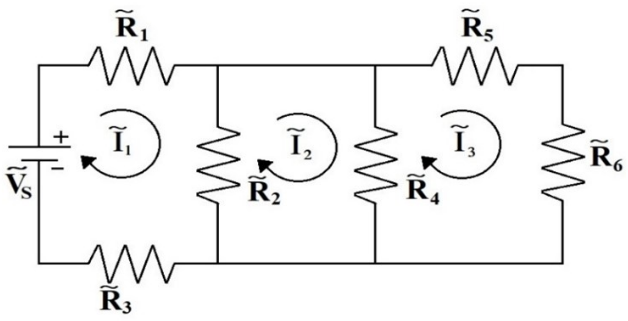

Figure 1. Electric Circuit

Three closed loops are present in the following Figure 1. The power source is $\widetilde{V}_S$, the resistors are $\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, \widetilde{R}_4, \widetilde{R}_5, \widetilde{R}_6$ and the current flows are $\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3$. We now apply Kirchhoff's law to each loop.

$$
\begin{aligned}
\widetilde{V_s} - \widetilde{I_1}\widetilde{R}_1 - \widetilde{I_1}\widetilde{R}_2 + \widetilde{I_2}\widetilde{R}_2 - \widetilde{I_1}\widetilde{R}_3 &= \widetilde{0} \\
-\widetilde{I_2}\widetilde{R}_4 + \widetilde{I_3}\widetilde{R}_4 - \widetilde{I_2}\widetilde{R}_2 + I_1\widetilde{R}_2 &= \widetilde{0} \\
-\widetilde{I_3}\widetilde{R}_5 - \widetilde{I_3}\widetilde{R}_6 - \widetilde{I_3}\widetilde{R}_4 + \widetilde{I_2}\widetilde{R}_4 &= \widetilde{0}
\end{aligned}
$$

Rearrange the above equations into matrix solutions.

$$
\begin{aligned}
(\widetilde{R}_1 + \widetilde{R}_2 + \widetilde{R}_3)\widetilde{I_1} - \widetilde{R}_2\widetilde{I_2} &= \widetilde{V_s} \\
-\widetilde{R}_2\widetilde{I_1} + (\widetilde{R}_2 + \widetilde{R}_4)\widetilde{I_2} - \widetilde{R}_4\widetilde{I_3} &= \widetilde{0} \\
-\widetilde{R}_4\widetilde{I_2} + (\widetilde{R}_4 + \widetilde{R}_5 + \widetilde{R}_6)\widetilde{I_3} &= \widetilde{0}
\end{aligned}
$$

And then write it in matrix form as follows

$$
\begin{bmatrix} \widetilde{R}_1 + \widetilde{R}_2 + \widetilde{R}_3 & -\widetilde{R}_2 & 0 \\ -\widetilde{R}_2 & \widetilde{R}_2 + \widetilde{R}_4 & -\widetilde{R}_4 \\ 0 & -\widetilde{R}_4 & \widetilde{R}_4 + \widetilde{R}_5 + \widetilde{R}_6 \end{bmatrix} \begin{bmatrix} \widetilde{I_1} \\ \widetilde{I_2} \\ \widetilde{I_3} \end{bmatrix} = \begin{bmatrix} \widetilde{e_1} \\ \widetilde{e_2} \\ \widetilde{e_3} \end{bmatrix}
$$

The solution to the above matrix equation is given by

$$
\widetilde{I} = \widetilde{R}^{-1}\widetilde{e} \tag{9}
$$

Calculating the inverse matrix using the proposed algorithm and multiplying it by $\tilde{e}$ will reveal the current flow. Consider the electrical circuit shown in Figure 1. Find the current in each mesh, given that

$$
\begin{aligned}
&\widetilde{R}_1 = [1.25, 2.75]\,\Omega, \\
&\widetilde{R}_2 = [4.51, 5.49]\,\Omega, \\
&\widetilde{R}_3 = [1.70, 2.30]\,\Omega, \\
&\widetilde{R}_4 = [3.52, 4.480]\,\Omega, \\
&\widetilde{R}_5 = [13.251, 16.749]\,\Omega, \\
&\widetilde{R}_6 = [4.51, 5.49]\,\Omega, \\
&\widetilde{V}_s = [113, 127]\,\text{V}.
\end{aligned}
$$

$$
\begin{bmatrix} [7.460, 10.540] & [-5.49, -4.51] & [0, 0] \\ [-5.49, -4.51] & [8.030, 9.970] & [-4.480, -3.52] \\ [0, 0] & [-4.480, -3.52] & [21.281, 26.719] \end{bmatrix} \begin{bmatrix} \hat{I}_1 \\ \hat{I}_2 \\ \hat{I}_3 \end{bmatrix} = \begin{bmatrix} [113, 127] \\ [0, 0] \\ [0, 0] \end{bmatrix}
$$

Applying the tridiagonal interval matrix inverse algorithm yields:

$$
\tilde{R}^{-1} = \begin{bmatrix} [0.120, 0.216] & [0.057, 0.147] & [0.009, 0.032] \\ [0.057, 0.148] & [0.134, 0.236] & [0.022, 0.053] \\ [0.007, 0.026] & [0.017, 0.043] & [0.040, 0.054] \end{bmatrix}
$$

The solution to the above matrix Equation (9) is

$$
\begin{bmatrix} \hat{I}_1 \\ \hat{I}_2 \\ \hat{I}_3 \end{bmatrix} = \begin{bmatrix} [0.120, 0.216] & [0.057, 0.147] & [0.009, 0.032] \\ [0.057, 0.148] & [0.134, 0.236] & [0.022, 0.053] \\ [0.007, 0.026] & [0.017, 0.043] & [0.040, 0.054] \end{bmatrix} \begin{bmatrix} [113, 127] \\ [0, 0] \\ [0, 0] \end{bmatrix}
$$

The current flow in each mesh of an electrical circuit

$$
\begin{aligned}
\widetilde{I}_1 &= [13.56, 26.76], \\
\widetilde{I}_2 &= [6.441, 18.159], \\
\widetilde{I}_3 &= [0.791, 3.169].
\end{aligned}
$$

5. CONCLUSION

This paper presents a novel method for computing the inverse of a general tridiagonal interval matrix using generalized interval arithmetic and LU decomposition. This approach can revolutionize electric circuit analysis by improving design, optimization, and reliability. This is done through accurate uncertainty representation and efficient inverse computation. Moreover, this methodology has broader applications in various domains where uncertainty modeling and precise calculations are crucial. Future research should optimize the algorithm for larger matrices, explore parallel computing techniques, and extend its application to more complex circuit topologies. This will enhance its practicality and impact.

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