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Research Paper

Results on Toeplitz Determinants for Subclasses of Analytic Functions Associated to q-Derivative Operator

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Abstract
An analytic function, also known as a holomorphic function, is a complex-valued function that is differentiable at every point within a given domain. In other words, a function $f(z)$ is analytic in a domain U if it has a derivative $f'(z)$ at every point z in U. Let A represent the set of functions f that are analytic within the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. These functions possess a normalized Taylor-Maclaurin series expansion written in the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ where $a_n \in \mathbb{C}, n = 2, 3, \ldots$. In recent years, the field of q-calculus has gained significant attention and research interest among mathematicians. The applications of this field are broadly applied in numerous subdivisions of physics and mathematics. In this research, we assume that S_q^* and $\mathbb{R}_q$ are subclasses of analytic functions obtained by applying the q-derivative operator. The objective of this paper is to obtain estimates for coefficient inequalities and Toeplitz determinants whose elements are the coefficients a_n for $f \in S_q^*$ and $f \in R_q$.

Keywords
Analytic Functions, Toeplitz Determinant, Quantum (or q-) Calculus, q-Derivative Operator

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1. INTRODUCTION

In the recent past, the field of q-calculus has become a trend and research interest among mathematicians. The use of this field is widely applied in innumerable subdivides of physics and mathematics. The implementation of q-calculus was established by (Jackson, 1909). He was the pioneer to evolve the q-integral and q-derivative in a coherent way. This study begins by letting A be the class of functions in the form of

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$D = \{z \in \mathbb{C}\, |z| < 1\}$$

For a function $f \in A$ given by (1.1) and $0 < q < 1$, the q-derivative operator of a function f is defined by (Jackson, 1909; Jackson, 1910) as

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & \text{for } z \neq 0 \\ f'(0), & \text{for } z = 0 \end{cases} \tag{1.2}$$

and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.2), it is obvious to show that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

with q-analogue of n is

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{i=0}^{n-1} q^i$$

If $q \to 1^-$ then $[n]_q \to n$.

Making use of the q-derivative operator of a function f, $D_q f(z)$, certain subclasses of A which are denoted by S_q^* and R_q are defined by the following:

$$S_q^* = \{f \in A\colon\ \mathrm{Re}\left(\frac{z(D_q f(z))}{f(z)}\right) > 0, z \in D\}$$

$$R_q = \{f \in A\colon\ \mathrm{Re}(D_q f(z)) > 0, z \in D\}$$

We note that S_q^* reduced to class S^* and R_q reduced to R as $q \to 1^-$.

In this study, we investigate the symmetric Toeplitz determinants for functions f which belongs to the classes S_q^* and

R_q. The Toeplitz determinants are closely interconnected to the Hankel determinants by Bansal (2013); Choo and Janteng (2013); Hern et al. (2020); Huey et al. (2023); Janteng et al. (2007); Karahuseyin et al. (2017); Sun et al. (2023). The symmetric Toeplitz determinants, $T_r(n)$, for analytic functions f in the form of (1.1) is established by Ali et al. (2018) as:

$$T_r(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_n & \cdots & a_{n+r-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r-2} & \cdots & a_n \end{vmatrix}$$

where $n, r = 1, 2, 3, \ldots$ with $a_1 = 1$. Specifically,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix},$$

$$T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},$$

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix},$$

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_3 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

The estimates for Toeplitz determinants $T_r\ (n)$ for functions in S_q^* and R, when n and r are small have been studied in Ali et al. (2018), Al-Khafaji et al. (2020), Al-shbeil et al. (2022), Buyankara and Çağlar (2023), Soh et al. (2021), Radhika et al. (2018), Ramachandran and Kavitha (2017), Ayinla and Bello (2021), Rasheed et al. (2023), Sivasubramanian et al. (2016), Srivastava et al. (2019), Tang et al. (2023), Tang et al. (2021), Wahid et al. (2022), Wanas et al. (2023). Motivated by these results, this study aims to find the determinants of Toeplitz determinants $T_r\ (n)$ for functions in S_q^* and R_q, when n and r are small.

2. PRELIMINARY RESULTS

To derive our results, we state the preliminary results which are required to prove the main results. Let $\mathcal{P}$ be the class of functions with positive real part containing all analytic functions $p : D \rightarrow C$ satisfying $p(0) = 1$ and $Re(p(z)) > 0$.

Lemma 1.1 (Duren, 1983) If the function $p \in \mathcal{P}$ is given by the series $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$, then the following sharp estimates holds

$$|c_n| \le 2, n = 1, 2, 3, \ldots$$

Lemma 1.2 (Efraimidis, 2016) Let $p \in \mathcal{P}$ and $\mu \in \mathcal{C}$. Then

$$|c_n - \mu c_k c_{n-k}| \le 2\max\{1, |2\mu - 1|\}, \quad 1 \le k \le n-1$$

The inequality is sharp for the $p(z) = (1 + z)/(1 - z)$ or its rotations if $|2\mu - 1| \ge 1$. While the inequality is sharp for the $p(z) = (1 + z^n)/(1 - z^n)$ or its rotations if $|2\mu - 1| < 1$.

3. MAIN RESULTS

Next, we enunciate our main results. The first result is the estimates for coefficients a_n for $f \in S_q^*$.

Theorem 2.1 Let $f \in S_q^*$ given by (1.1), then for $n \ge 1$

$$|a_n| \le \frac{2}{[n]_q - 1} \prod_{j=2}^{n-1} \frac{[j]_q + 1}{[j]_q - 1} \tag{2.1}$$

Proof. Since $f \in S_q^*$, it follows that

$$z(D_q f(z)) = f(z)p(z)$$

and by equating the coefficients, we obtain

$$a_2 = \frac{c_1}{[2]_q - 1} \tag{2.2}$$

$$a_3 = \frac{c_2}{[3]_q - 1} + \frac{c_1^2}{[2]_q - 1[3]_q - 1} \tag{2.3}$$

$$\begin{aligned} a_4 = &\frac{c_1 c_2}{[2]_q - 1[4]_q - 1} + \frac{c_1 c_2}{[3]_q - 1[4]_q - 1} \\ &+ \frac{c_1^3}{[2]_q - 1[3]_q - 1[4]_q - 1} + \frac{c_3}{[4]_q - 1} \end{aligned} \tag{2.4}$$

By using Lemma 1.1 in (2.2), (2.3) and (2.4), we get

$$|a_2| \le \frac{2}{[2]_q - 1},$$
$$|a_3| \le \frac{2([2]_q + 1)}{[2]_q - 1[3]_q - 1}.$$

and

$$|a_4| \le \frac{2([2]_q + 1)([3]_q + 1)}{([2]_q - 1)([3]_q - 1)([4]_q - 1)}$$

It follows that (2.1) holds for $n = 2, 3, 4$. We now prove (2.1) by using induction.

By equating $z(D_q f(z)) = f(z)p(z)$ for z^n, we have

$$[n]_q a_n = a_n + c a_{n-1} + c a_{n-2} + \cdots + c_{n-2} a_2 + c_{n-1} \tag{2.5}$$

Conjointly with Lemma 1.1, (2.5) will yield to

$$|a_n| \leq \frac{2}{[n]_q - 1}\left[1 + \sum_{k=2}^{n-1} |a_k|\right] \tag{2.6}$$

We presume that (2.1) holds for $k = 2, 3, 4, \ldots, n-1$. Then from (2.6) and Theorem 2.1, we obtain

$$|a_n| \leq \frac{2}{[n]_q - 1}\left[1 + \sum_{k=2}^{n-1} \frac{2}{[k]_q - 1} \prod_{j=2}^{k-1} \frac{[j]_q + 1}{[j]_q - 1}\right] \tag{2.7}$$

As a way to conclude the proof, it is adequate to show that

$$\frac{2}{[m]_q - 1}\left[1 + \sum_{k=2}^{m-1} \frac{2}{[k]_q - 1} \prod_{j=2}^{k-1} \frac{[j]_q + 1}{[j]_q - 1}\right] = \frac{2}{[m]_q - 1} \prod_{j=2}^{m-1} \frac{[j]_q + 1}{[j]_q - 1} \tag{2.8}$$

(2.8) is valid for $n = 2, 3, 4$. Let us suppose that (2.8) is true for all $m = 2, 3, 4, \ldots, n-1$.
Then from (2.7), we note that

$$\frac{2}{[n]_q - 1}\left[1 + \sum_{k=2}^{n-1} \frac{2}{[k]_q - 1} \prod_{j=2}^{k-1} \frac{[j]_q + 1}{[j]_q - 1}\right]$$

$$\left(\frac{[n-1]_q - 1}{[n]_q - 1}\right)\left(\frac{2}{[n-1]_q - 1}\right)\left[1 + \sum_{k=2}^{n-2} \frac{2}{[k]_q - 1} \prod_{j=2}^{k-1} \frac{[j]_q + 1}{[j]_q - 1}\right] + \left(\frac{2}{[n]_q - 1}\right)\left(\frac{2}{[n-1]_q - 1}\right)\prod_{j=2}^{n-2} \frac{[j]_q - 1}{[j]_q - 1}$$

$$\begin{aligned}
&\left(\frac{[n-1]_q - 1}{[n]_q - 1}\right)\frac{2}{[n-1]_q - 1}\prod_{j=2}^{n-2} \frac{[j]_q + 1}{[j]_q - 1} \\
&+ \left(\frac{2}{[n]_q - 1}\right)\left(\frac{2}{[n-1]_q - 1}\right)\prod_{j=2}^{n-2} \frac{[j]_q + 1}{[j]_q - 1} \\
&= \frac{2}{[n-1]_q - 1}\prod_{j=2}^{n-2}\left(\frac{[j]_q + 1}{[j]_q - 1}\right)\left(\frac{[n-1]_q + 1}{[n]_q - 1}\right) \\
&= \frac{2}{[n]_q - 1}\prod_{j=2}^{n-1} \frac{[j]_q + 1}{[j]_q - 1}.
\end{aligned}$$

Thus, (2.8) holds for $m = n$ and hence (2.1) holds.

Next, we give the estimates for Toeplitz determinants whose elements are the coefficients a_n for $f \in S_q^*$.

Theorem 2.2 Let $f \in S_q^*$ be in the form (1.1). Then,

(i) $$|T_2(n)| \leq \frac{4}{[n]_q - 1^2}\prod_{j=2}^{n-1}\frac{([j]_q + 1)^2}{([j]_q - 1)^2} \times \left[1 + \frac{([n]_q + 1)^2}{([n]_q - 1)^2}\right], \text{ for } n \geq 2$$

(ii) $$|T_3(1)| \leq \left[1 + \frac{2([2]_q + 1)}{([2]_q - 1)([3]_q - 1)}\right] \times \left[1 + \frac{8}{([2]_q - 1)^2} - \frac{2([2]_q + 1)}{([2]_q - 1)([3]_q - 1)}\right]$$

(iii) $$|T_3(2)| \leq \frac{8}{([2]_q - 1)^3}\left[1 + \frac{([2]_q + 1)([3]_q + 1)}{([3]_q - 1)([4]_q - 1)}\right] \times \left[1 + \frac{2([2]_q + 1)^2}{([3]_q - 1)^2} - \frac{([2]_q + 1)([3]_q + 1)}{([3]_q - 1)([4]_q - 1)}\right]$$

Proof. Let $f \in S_q^*$ be in the form (1.1). Then with Theorem 2.1,

$$\begin{aligned}
|T_2(n)| &= |a_n^2 - a_{n+1}^2| \\
&\leq |a_n^2| + |a_{n+1}^2| \\
&\leq \left[\frac{2}{[n]_q - 1}\prod_{j=2}^{n-1}\left(\frac{[j]_q + 1}{[j]_q - 1}\right)\right]^2 \\
&\quad + \left[\frac{2}{[n+1]_q - 1}\prod_{j=2}^{n}\left(\frac{[j]_q + 1}{[j]_q - 1}\right)\right]^2 \\
&= \frac{4}{[n]_q - 1^2}\prod_{j=2}^{n-1}\frac{([j]_q + 1)^2}{([j]_q - 1)^2}\left[1 + \frac{([n]_q + 1)^2}{([n]_q - 1)^2}\right]
\end{aligned}$$

Next, if $f \in S_q^*$ if of the form (1.1), then

$$|T_3(1)| = |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2|$$

By applying the triangle inequalities, Theorem 2.1 and Lemma 1.2 in S_q^*, we have

$$\begin{aligned}
|T_3(1)| &\leq 1 + 2|a_2^2| + |a_3||a_3 - 2a_3^2| \\
&\leq 1 + \frac{8}{[2]_q - 1^2} + \frac{2([2]_q + 1)}{([2]_q - 1)([3]_q - 1)} \\
&\times \left[\frac{8}{[2]_q - 1^2} - \frac{2([2]_q + 1)}{([2]_q - 1)([3]_q - 1)}\right] \\
&= \left[1 + \frac{2([2]_q + 1)}{([2]_q - 1)([3]_q - 1)}\right] \\
&\times \left[1 + \frac{8}{[2]_q - 1^2} + \frac{2([2]_q + 1)}{([2]_q - 1)([3]_q - 1)}\right]
\end{aligned}$$

Eventually, if $f \in S_q^*$ is of the form (1.1), then

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 + a_2a_4 - 2a_3^2)|$$

By the triangle inequalities, we have

$$|T_3(2)| \leq |a_2 - a_4||a_2^2 + a_2a_4 - 2a_3^2|$$

In Theorem 2.1, clearly that

$$|a_2 - a_4| \leq |a_2| + |a_4|$$
$$= \frac{2}{[2]_q - 1}\left[1 + \frac{([2]_q + 1)([3]_q + 1)}{([3]_q - 1)([4]_q - 1)}\right]$$

Thus, we need to maximize $|a_2^2 + a_2a_4 - 2a_3^2|$ for functions in S_q^*. Therefore, $|a_2^2 + a_2a_4 - 2a_3^2|$

$$= \left|\frac{(c_1^2)}{([2]_q - 1)^2} + \frac{1}{([2]_q - 1)([3]_q - 1)([4]_q - 1)} - \frac{2}{([2]_q - 1)^2([3]_q - 1)^2}c - \frac{(2c_2^2)}{([3]_q - 1)^2} + \frac{1}{([2]_q - 1)([3]_q - 1)([4]_q - 1)} + \frac{1}{([2]_q - 1)^2([4]_q - 1)} - \frac{4}{([2]_q - 1)([3]_q - 1)^2}c_1^2c_2 + \frac{(c_1c_3)}{([2]_q - 1)([4]_q - 1)}\right|$$

By Lemma 1.1 and Lemma 1.2, it will yield to $|a_2^2 + a_2a_4 - 2a_3^2|$

$$\leq \frac{4}{[2]_q - 1} + \frac{16}{([2]_q - 1)^2([3]_q - 1)}\left(\frac{2}{[2]_q - 1} - \frac{1}{[4]_q - 1}\right) + \frac{8}{[3]_q - 1^2} + \frac{2}{[2]_q - 1([4]_q - 1)} \times \left(\frac{16([4]_q - 1)}{[3]_q - 1^2} - \frac{4}{[2]_q - 1} - \frac{4}{[2]_q - 1} - 2\right)$$
$$= \frac{4}{[2]_q - 1^2}\left[1 + \frac{2([2]_q + 1)^2}{[3]_q - 1^2} - \frac{([2]_q + 1)([3]_q + 1)}{([3]_q - 1)([4]_q - 1)}\right]$$

Therefore,

$$|T_3(2)| \leq \frac{8}{[2]_q - 1^3}\left[1 + \frac{([2]_q + 1)([3]_q + 1)}{([3]_q - 1)([4]_q - 1)}\right] \times \left[1 + \frac{2([2]_q + 1)^2}{[3]_q - 1^2} - \frac{([2]_q + 1)([3]_q + 1)}{([3]_q - 1)([4]_q - 1)}\right]$$

Remark 2.1 For $n = 2$ and $n = 3$ in Theorem 2.2 (i), we get the

$$|T_2(2)| \leq \frac{4}{[2]_q - 1^2}\left[1 + \frac{([2]_q + 1)^2}{[3]_q - 1^2}\right]$$

and

$$|T_2(3)| \leq \frac{4([2]_q + 1)^2}{([2]_q - 1)^2([3]_q - 1)^2}\left[1 + \frac{([3]_q + 1)^2}{([4]_q - 1)^2}\right]$$

Corollary 2.1 (Ali et al., 2018) By taking $q = 1$ in Theorem 2.2 for $f \in S_q^*$ given by (1.1), then $|T_2(2)| \leq 2n^2 + n + 1$, for $n \geq 2$, $|T_3(1)| \leq 24$ and $|T_3(2)| \leq 84$. All the inequalities are sharp.

Next, by using the same method with Theorem 2.1, we can prove the coefficients inequalities for the class R_q. Thus, we omit the proofs.

Theorem 2.3 Let $f \in R_q$ given by (1.1), then for $n \geq 2$

$$|a_n| \geq \frac{2}{[n]_q}$$

Finally, we can prove the Toeplitz determinant for the class R_q as follows.
Theorem 2.4 Let $f \in R_q$ given by (1.1), then

(i) $|T_2(n)| \leq \dfrac{4}{[n]_q{}^2} + \dfrac{4}{[n+1]_q{}^2}$, for $n \geq 2$

(ii) $|T_3(1)| \leq \left[1 + \dfrac{2}{[3]_q}\right]\left[1 - \dfrac{2}{[3]_q} + \dfrac{8}{[2]_q{}^2}\right]$

Remark 2.3 For $n = 2$ and $n = 3$ in Theorem 2.4 (i), we get the

$$|T_2(2)| \leq \frac{4}{[2]_q{}^2} + \frac{4}{[3]_q{}^2}$$

and

$$|T_2(3)| \leq \frac{4}{[3]_q{}^2} + \frac{4}{[4]_q{}^2}$$

Corollary 2.3 (Ali et al., 2018) By taking $q = 1$ in Theorem 2.4 for $f \in R_q$ given by (1.1), then $|T_2(2)| \leq 4/n^2 + 4/(n+1)^2$, for $n \geq 2$ and $|T_3(1)| \leq 35/9$. All the inequalities are sharp.

4. CONCLUSIONS

In conclusion, we obtained the estimates for coefficient inequalities and Toeplitz determinants whose elements are the coefficients a_n for $f \in S_q^*$ and $f \in R_q$.

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