Science and Technology Indonesia
e-ISSN:2580-4391 p-ISSN:2580-4405
Vol. 9, No. 3, July 2024

Research Paper

Generalized Hölder Inequality in Herz-Morrey Spaces with Variable Exponent

Hairur Rahman[1]*, Corina Karim[2]

[1]Departement of Mathematics, Islamic State University Maulana Malik Ibrahim Malang, Malang, 65144, Indonesia
[2]Department of Mathematics, Universitas Brawijaya, Malang, 65145, Indonesia
*Corresponding author: hairur@mat.uin-malang.ac.id

Abstract
The paper investigates the conditions for the generalized Hölder's inequality with a variable exponent in Herz-Morrey spaces. The main results are based on the exponent functions $p(\cdot)$ and $\alpha(\cdot)$. The proof of the first main result using the generalized Hölder's inequality in Lebesgue spaces. The second main result of the paper is related to the weak space of the generalized Hölder's inequality with a variable exponent in Herz-Morrey spaces. The theorems state the equivalence of certain conditions for the inequality. Mathematical proofs and analysis are providing to support the presented results for findings contribute to the understanding of Hölder's inequalities in variable exponent spaces and their applications in Herz-Morrey spaces.

Keywords
Hölder Inequality, Herz-Morrey Spaces, Variable Exponent

Received: 6 July 2023, Accepted: 18 May 2024
https://doi.org/10.26554/sti.2024.9.3.637-641

1. INTRODUCTION

Variable exponent was initially examined in 1931 in the context of Lebesgue spaces by Orlicz (1931). Over the past thirty years, there has been a growing interest in the variable exponent, particularly in its applications in various fields such as electrorheological fluid dynamics, differential equations, among others. The concept of variable exponent led to the development of function spaces with variable exponents including Lebesgue spaces (Aoyama, 2009), Morrey spaces (Almeida et al., 2008; Ifronika et al., 2017; Wajih and Gunawan, 2020), Morrey and Campanato spaces (Fan, 2010), Hardy spaces (Nakai and Sawano, 2012), and Bessel potential spaces (Gurka et al., 2007).

In 2015, Xu and Yang (2015) conducted a study on Herz-Morrey spaces with variable exponents. Consider a scenario where $0 < q < \infty$, $p(\cdot) \in (P\mathbb{R}^n)$, and $0 \leq \lambda \leq \infty$. Suppose that represents a bounded real-valued measurable function on. The space denoted as $MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ known as the homogeneous Herz-Morrey space, is formally defined as

$$MK^{\alpha(\cdot),\lambda}_{p(\cdot),q} := \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} < \infty \right\}$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} \|2^{\alpha(\cdot)k} f\chi_k\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}$$

with $B_k = B(0.2^k) = \left\{ x \in \mathbb{R}^n : |x| \leq 2^k \right\}$ where $A_k = B_k \backslash B_{k-1}$ for all $k \in \mathbb{Z}$, and $\chi_k = \chi_{A_k}$ denote the characteristic function of the set A_k for all $k \in \mathbb{Z}$.

Given that each function space possesses its own associated weak spaces, the weak Herz-Morrey spaces with a variable exponent $MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ are established through adjustments to the original definition (Shanzhen and Lifang, 2005).

Definition 1. For $0 < q \leq \infty$, $\alpha(\cdot) \in \mathbb{R}^n$ and $p(\cdot) \in P\mathbb{R}^n$,

$$_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q} := \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} < \infty \right\}$$

where

$$\|f\|_{_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}(\mathbb{R}^n)} = \sup_{\alpha\in\mathbb{R}^n} \gamma \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p(\cdot)} m_k(\gamma, f)^{\frac{p(\cdot)}{q}} \right)^{\frac{1}{p(\cdot)}}$$

Some researchers have conducted research on the Herz-Morrey space (Chen et al., 2014; Dung et al., 2023; Izuki, 2010; Mizuta, 2016; Mizuta and Ohno, 2015; Nogayama, 2019; Sultan et al., 2022; Wang and Liao, 2020; Wang and Wu, 2016). Nevertheless, the Hölder inequality also plays a significant role in this space (Alshanti et al., 2023; Ifronika et al., 2018; Matkowski, 2009; Vasyunin, 2004). In the year 2017 and 2018, Masta et al. (2017, 2018) identified the adequate and indispensable conditions for the generalized Hölder inequality in Lebesgue spaces.

Within this manuscript, the generalized Hölder inequality in Herz-Morrey spaces with a variable exponent can be elaborated. The research on the generalized Hölder inequality in Herz-Morrey spaces with variable exponent delves into fundamental functional analysis inequalities. This comprehensive investigation contributes significantly to understanding inequalities in variable exponent function spaces.

2. EXPERIMENTAL SECTION

The experimental section is focused on theoretical analysis and mathematical proofs rather than empirical experiments, illustrating the conditions and equivalence of Hölder inequality in variable exponent spaces within Herz-Morrey spaces. The derivation and validation of the results within Herz-Morrey spaces with variable exponents are conducted through theoretical analysis. Furthermore, mathematical proofs are provided to support the main results, demonstrating the equivalence of certain conditions for the inequality in variable exponent spaces.

3. RESULTS AND DISCUSSION

In this section, we aim to introduce a theorem and lemma that have been formulated based on the essential conditions required for Hölder's inequality with a variable exponent within Herz-Morrey spaces. The first main results, for the generalized Hölder's inequality with variable exponent in Herz-Morrey spaces, is the following:

Theorem 1. Let $m \geq 2$. If $\alpha(\cdot), \alpha_i(\cdot) \in \mathbb{R}^n, 0 \leq \lambda, \lambda_i < \infty, p(\cdot), p_i(\cdot) \in P\mathbb{R}^n$, and $0 < q, q_i < \infty$, for each $i = 1, \ldots, m$ then the subsequent statements are equivalent:

1. $\sum_{i=1}^{m} \frac{1}{\alpha_i(\cdot)} \leq \frac{1}{\alpha(\cdot)}, \sum_{i=1}^{m} \lambda = \lambda_i, \sum_{i=1}^{m} \frac{1}{p_i(\cdot)} \leq \frac{1}{p(\cdot)}$, and $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$.
2. $\|\prod_{i=1}^{m} f_i\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leq \prod_{i=1}^{m} \|f_i\|_{MK^{\alpha(\cdot),\lambda_i}_{p(\cdot),q_i}}$, for every $f_i \in MK^{\alpha(\cdot),\lambda_i}_{p(\cdot),q_i}\mathbb{R}^n, i = 1, \ldots, m$.

Proof. ($1 \rightarrow 2$) Let $\sum_{i=1}^{m} \frac{1}{\alpha_i(\cdot)} \leq \frac{1}{\alpha(\cdot)}, \sum_{i=1}^{m} \lambda = \lambda_i, \sum_{i=1}^{m} \frac{1}{p_i(\cdot)} \leq \frac{1}{p(\cdot)}$, and $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$ hold. Suppose that $\frac{1}{\alpha^*(\cdot)} := \sum_{i=1}^{m} \frac{1}{\alpha(\cdot)}$ Clearly we have $\alpha^*(\cdot) \geq \alpha(\cdot)$. Now take $f_i \in MK^{\alpha(\cdot),\lambda_i}_{p(\cdot),q_i}\mathbb{R}^n$, where $i = 1, \ldots, m$. By utilizing the generalized Hölder's inequality within the framework of Lebesgue spaces, it can be demonstrated that

$$\begin{aligned}
\left\|\prod_{i=1}^{m} f_i\right\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{\alpha(\cdot)k} \left\|\prod_{i=1}^{m} f_i\chi_k\right\|^{q}_{L^{p(\cdot)}\mathbb{R}^n}\right)^{\frac{1}{q}} \\
&\leq \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{\alpha^*(\cdot)k} \left\|\prod_{i=1}^{m} f_i\chi_k\right\|^{q^*}_{L^{p(\cdot)}\mathbb{R}^n}\right)^{\frac{1}{q}} \\
&\leq \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{\alpha^*(\cdot)k} \prod_{i=1}^{m} \|f_i\chi_k\|^{q^*}_{L^{p(\cdot)}\mathbb{R}^n}\right)^{\frac{1}{q}} \\
&\leq \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{\alpha^*(\cdot)k} \prod_{i=1}^{m} \|f_i\chi_k\|^{q^i}_{L^{p(\cdot)}\mathbb{R}^n}\right)^{\frac{1}{q}} \\
&= \prod_{i=1}^{m} \|f_i\chi_k\|_{MK^{\alpha(\cdot),\lambda_i}_{p(\cdot),q_i}}
\end{aligned}$$

Taking the supremum of $2^{-L\lambda}$ we obtain

$$\left\|\prod_{i=1}^{m} f_i\right\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} \leq \prod_{i=1}^{m} \|f_i\chi_k\|_{MK^{\alpha(\cdot),\lambda_i}_{p(\cdot),q_i}}$$

($2 \leftarrow 1$). Suppose that $\|\prod_{i=1}^{m} f_i\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leq \prod_{i=1}^{m} \|f_i\chi_k\|_{MK^{\alpha(\cdot),\lambda_i}_{p(\cdot),q_i}}$. For every $f_i \in MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n, \lambda\mathbb{R}^n, i = 1, \ldots, m$. Choose $f_i := \chi_{B(0,R)}$. It can be deduced from the hypothesis that

$$\begin{aligned}
\|\chi_{B(0,R)}\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} = \left\|\prod_{i=1}^{m} f_i\right\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} &\leq \prod_{i=1}^{m} \|f_i\chi_k\|_{MK^{\alpha_i(\cdot),\lambda_i}_{p_i(\cdot),q_i}} \\
&= \prod_{i=1}^{m} \|\chi_{B(0,R)}\|_{MK^{\alpha_i(\cdot),\lambda}_{p_i(\cdot),q}}
\end{aligned}$$

Next, by choosing $0 < \epsilon < \min\left\{\frac{n\alpha_1(\cdot)}{p_1(\cdot)}, \ldots, \frac{n\alpha_m(\cdot)}{p_m(\cdot)}\right\}$. Clearly, $\epsilon < d$ and for arbitrary $K \in \mathbb{N}$, write $g_{\epsilon,K}(x) := \chi_{\{0\leq|x|<1\}} + \sum_{j=1}^{k} \chi_{\{j\leq j+j^{-\epsilon}\}}(x)$ (if desired, this can be simplified to the situation where the $d = 1$, and for the general case followed by an examination of the tensor product $g_{\epsilon,K}(x_1, x_2, \ldots, x_n) = g_{\epsilon,K}(x_1), g_{\epsilon,K}(x_2) \ldots, g_{\epsilon,K}(x_n)$, working with cubes stead of balls).

By defining $f_i := g_{\epsilon,K}, i = 1, \ldots, m$.

If $|\prod_{i=1}^{m} f_i\chi_k|^{p(\cdot)} = g_{\epsilon,K}$ then we have $\|\prod_{i=1}^{m} f_i\chi_k\|^{p(\cdot)} = g_{\epsilon,K}$.

Hence we obtain

$$\left\|\prod_{i=1}^{m} f_i\right\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{\alpha(\cdot)k} \left\|\prod_{i=1}^{m} f_i \chi_k\right\|^q_{L^{p(\cdot)}\mathbb{R}^n}\right)^{\frac{1}{q}}$$

$$= \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{\alpha(\cdot)k} \left(\int_{B(a,K+K^{\epsilon})} g_{\epsilon,K}(x)dx\right)^{\frac{q}{p(\cdot)}}\right)^{\frac{1}{q}}$$

$$\geq \left(\sum_{k=0}^{\infty} 2^{\alpha(\cdot)kq}\right)^{\frac{1}{q}}\left(\int_{B(a,K+K^{\epsilon})} g_{\epsilon,K}(x)dx\right)^{\frac{1}{p(\cdot)}}$$

$$\geq C\left(|B(0,1)| + \sum_{k=0}^{\infty}\int_{j\leq|x|\leq j+j^{-\epsilon}} g_{\epsilon,K}dx\right)^{\frac{1}{p(\cdot)}}$$

$$\geq C\left(|B(0,1)| + \sum_{k=0}^{\infty}(|B(0,j+j^{-\epsilon})| - (B(0,j)|)\right)^{\frac{1}{p(\cdot)}}$$

$$\geq C\left(\sum_{k=0}^{\infty}((j+j^{-\epsilon}) - j^d)\right)^{\frac{1}{p(\cdot)}}$$

$$\geq C(K+K^{-\epsilon})^{\frac{n}{p(\cdot)}-\frac{\epsilon}{p(\cdot)}}$$

This inequality also valid for the cases where $K = 1$, as long as the condition $C \leq 1$ is satisfied. This acquires the following result

$$\left\|\prod_{i=1}^{m} f_i\right\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} \geq C(K+K^{-\epsilon})^{\frac{n}{p(\cdot)}-\frac{\epsilon}{p(\cdot)}} (K+K^{-\epsilon})^{\frac{n}{p(\cdot)}-\frac{\epsilon}{p(\cdot)}} = (K+K^{-\epsilon})^{\frac{n}{p(\cdot)}-\frac{\epsilon}{p(\cdot)}}$$

For each integer $i = 1, \ldots, m$ assert that

$$\sup_{a\in\mathbb{R}^n,r>0} |B(a,r)|^{\frac{1}{p_i(\cdot)}-\frac{i}{\alpha_i(\cdot)}}\left(\int_{B(a,r)} f_i(x)^{\alpha_i x}dx\right)^{\frac{1}{\alpha_i(\cdot)}} \leq |B(0,L)|^{\frac{1}{p_i(\cdot)}-\frac{i}{\alpha_i(\cdot)}}\left(\int_{B(0,L)} f_i(x)^{\alpha_i x}dx\right)^{\frac{1}{\alpha_i(\cdot)}}$$

for a specific integer L with $2 \leq L \leq K+1$, it is worth mentioning that function $f_i = g_{\epsilon,K}$ demonstrates symmetry around the origin and centralizes the majority of its distribution near 0. Consequently, for every $\alpha \in \mathbb{R}^n$ and $r > 0$, it follows that

$$|B(a,r)|^{\frac{1}{p_i(\cdot)}-\frac{i}{\alpha_i(\cdot)}}\left(\int_{B(a,r)} f_i(x)^{\alpha_i x}dx\right)^{\frac{1}{\alpha_i(\cdot)}} \leq |B(0,r)|^{\frac{1}{p_i(\cdot)}-\frac{i}{\alpha_i(\cdot)}}\left(\int_{B(0,r)} f_i(x)^{\alpha_i x}dx\right)^{\frac{1}{\alpha_i(\cdot)}}$$

Indeed, the value of the bracket on the right-hand side, expressed only as a function of r , demonstrates a continuous increase as r ranges from 0 to 2, contrasting with a decrease for $r > K + K^{-\epsilon}$. This observation validates the assertion regarding the supremum. since $\frac{1}{p_i(\cdot)} - \frac{1}{\alpha_i(\cdot)} \leq 0$ for $i = 1, \ldots, m$ and $j + j^{-\epsilon} \leq 2j$ for $j = 1, \ldots, K$, it can be deduced that

$$\|f_i\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} = \sup_{a\in\mathbb{R}^n,r>0} |B(a,r)|^{\frac{1}{p_i(\cdot)}-\frac{i}{\alpha_i(\cdot)}}\left(\int_{B(a,r)} f_i(x)^{\alpha_i x}dx\right)^{\frac{1}{\alpha_i(\cdot)}}$$

$$\leq |B(0,L)|^{\frac{1}{p_i(\cdot)}-\frac{i}{\alpha_i(\cdot)}}\left(\int_{B(0,L)} f_i(x)^{\alpha_i x}dx\right)^{\frac{1}{\alpha_i(\cdot)}}$$

$$\leq CL^{\frac{n}{p_i(\cdot)}-\frac{n}{\alpha_i(\cdot)}}\left(|B(0,1)| + \sum_{j=1}^{L}((j+j^{\epsilon})^d - j^d)\right)^{\frac{1}{\alpha_i(\cdot)}}$$

$$\leq CL^{\frac{n}{p_i(\cdot)}-\frac{n}{\alpha_i(\cdot)}}\left(|B(0,1)| + \sum_{j=1}^{L} j^{d-\epsilon-1}\right)^{\frac{1}{\alpha_i(\cdot)}}$$

$$\leq CL^{\frac{n}{p_i(\cdot)}-\frac{n}{\alpha_i(\cdot)}} L^{\frac{n}{p_i(\cdot)}-\frac{n}{\alpha_i(\cdot)}}$$

$$= CL^{\frac{n}{p_i(\cdot)}-\frac{n}{\alpha_i(\cdot)}}$$

Furthermore, given $L \leq K+1 \leq 2(K+K^{\epsilon})$, it can be desired that
$\|f_i\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} \leq C(K+K^{\epsilon})^{\frac{n}{p_i(\cdot)}-\frac{n}{\alpha_i(\cdot)}}$ for $i = 1, \ldots, m$.
Infact $\sum_{j=1}^{m}\frac{n}{p_i(\cdot)} = \frac{n}{\alpha_i(\cdot)}$ and $\|\prod_{i=1}^{m} f_i\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \leq \|\prod_{i=1}^{m} f_i\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}$, we we conclude from the two inequalities above that $(K + K^{\epsilon})^{-\frac{\epsilon}{\alpha(\cdot)}+\sum_{j=1}^{m}\frac{\epsilon}{\alpha_i(\cdot)}} \leq C$ for every $K \in \mathbb{N}$ therefore $\sum_{j=1}^{m}\frac{\epsilon}{\alpha_i(\cdot)} \leq \frac{\epsilon}{\alpha(\cdot)}$ as desired.
Remark 2.2 For the cases where $m = 2$, the proof Theorem 2 below is derived.

Theorem 2. Let $\alpha(\cdot), \alpha_1(\cdot), \alpha_2(\cdot) \in \mathbb{R}^n, 0 \le \lambda, \lambda_1, \lambda_3 < \infty, p(\cdot), p_1(\cdot), p_2(\cdot) \in P\mathbb{R}^n$, and $0 < q, q_1, q_2 < \infty$. Then the subsequent statements are equivalent:

1. $\alpha_1(\cdot)+\alpha_2(\cdot) \le \alpha(\cdot), \lambda_1+\lambda_2 = \lambda, \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} \le \frac{1}{p(\cdot)}$, and $\frac{1}{q_1} + \frac{1}{q_2} \le \frac{1}{q}$.
2. $\|f_g\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \le \|f\|_{MK^{\alpha_1(\cdot),\lambda_1}_{p_1(\cdot),q_1}} \|g\|_{MK^{\alpha_2(\cdot),\lambda_2}_{p_2(\cdot),q_2}}$ for every $f \in MK^{\alpha_1(\cdot),\lambda_1}_{p_1(\cdot),q_1}\mathbb{R}^n$ and $g \in MK^{\alpha_2(\cdot),\lambda_2}_{p_2(\cdot),q_2}\mathbb{R}^n$

The relation between $MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n$ and ${}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n$ can be rewrite as the following Lemma:

Lemma 1. Let $0 < q \le \infty, 0 \le \lambda, \le \infty, \alpha(\cdot) \in \mathbb{R}^n$, and $p(\cdot) \in P\mathbb{R}^n$. Then $MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n \subseteq_w MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n\ with \|f\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n} \le \|f\|_{MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n}$ for every $f \in MK^{\alpha(\cdot),\lambda}_{p(\cdot),q}\mathbb{R}^n$

Meanwhile, our second primary finding pertains to the frail space of the generalized Hölder's inequality with a variable exponent within Herz-Morrey spaces.

Theorem 3. Let $m \ge 2$. If $\alpha(\cdot), \alpha_i(\cdot) \in \mathbb{R}^n, 0 \le \lambda, \lambda_i <\in p(\cdot), p_i(\cdot) \in P\mathbb{R}^n$, and $0 \le q, q_i < \infty$, for each $i = 1, \ldots, m$, then then the subsequent statements are equivalent:

1. $\sum_{i=1}^m \frac{1}{\alpha_i(\cdot)} \le \frac{1}{\alpha(\cdot)}, \sum_{i=1}^m \lambda = \lambda_i, \sum_{i=1}^m \frac{1}{p_i(\cdot)} \le \frac{1}{p(\cdot)}$, and $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$.
2. $\left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \le \prod_{i=1}^m \|f_i\|_{MK^{\alpha_i(\cdot),\lambda_i}_{p_i(\cdot),q_i}}$, for every $f_i \in MK^{\alpha_i(\cdot),\lambda_i}_{p_i(\cdot),q_i}\mathbb{R}^n, i = 1, \ldots, m$.

Proof. (1→2) Let $\sum_{i=1}^m \frac{1}{\alpha_i(\cdot)} \le \frac{1}{\alpha(\cdot)}, \sum_{i=1}^m \lambda = \lambda_i, \sum_{i=1}^m \frac{1}{p_i(\cdot)} \le \frac{1}{p(\cdot)}$, and $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$ hold. Suppose that $\frac{1}{\alpha^*(\cdot)} := \frac{1}{\alpha_i(\cdot)}$ and $\frac{1}{p^*(\cdot)} := \frac{1}{p_i(\cdot)}$. Clearly we have $\alpha^*(\cdot) \ge \alpha(\cdot)$ and $p^*(\cdot) \ge p(\cdot)$. Taking $f_i \in {}_wMK^{\alpha_i(\cdot),\lambda_i}_{p_i(\cdot),q_i}\mathbb{R}^n, i = 1, \ldots, m$ and by using the generalized Hlder's inequality in Lebesgue spaces, it can be demonstrated that

$$\begin{aligned}\left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} &= \sup_{a\mathbb{R}^n} \sup_{L\in\mathbb{Z}} \gamma\, 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p(\cdot)} m_k \left(\gamma, \left\|\prod_{i=1}^m f_i\right\|\right)^{\frac{p^*(\cdot)}{q}}\right)^{\frac{1}{p^*(\cdot)}} \\ &\le \sup_{a\mathbb{R}^n} \sup_{L\in\mathbb{Z}} \gamma\, 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha^*(\cdot)p^*(\cdot)} m_k \left(\gamma, \left\|\prod_{i=1}^m f_i\right\|\right)^{\frac{p^*(\cdot)}{q}}\right)^{\frac{1}{p^*(\cdot)}} \\ &\le \sup_{a\mathbb{R}^n} \sup_{L\in\mathbb{Z}} \gamma\, 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha^i(\cdot)p^i(\cdot)} m_k \left(\gamma, \left\|\prod_{i=1}^m f_i\right\|\right)^{\frac{p^*(\cdot)}{q}}\right)^{\frac{1}{p^*(\cdot)}} \\ &= \left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha_i(\cdot),\lambda_i}_{p_i(\cdot),q_i}(\mathbb{R}^n)}\end{aligned}$$

Taking supremium $2^{-L\lambda}$, we obtain

$$\left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}(\mathbb{R}^n)} \le \left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha_i(\cdot),\lambda_i}_{p_i(\cdot),q_i}(\mathbb{R}^n)}$$

(2→1). Let $R\ >)$ arbitary and $f_i := \chi_{B(0,R)}$ for each $i = 1, \ldots, m$. According to the hypothesis, we are able to derive

$$\begin{aligned}\|\chi_{B(0,R}\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} = \left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} &\le m\prod_{i=1}^m \|f_i\chi_k\|_{{}_wMK^{\alpha_i(\cdot),\lambda}_{p_i(\cdot),q}} \\ &= \prod_{i=1}^m \|\chi_{B(0,R)}\|_{MK^{\alpha_i(\cdot),\lambda}_{p_i(\cdot),q}}\end{aligned}$$

Hence $R^{\frac{n}{p(\cdot)}\sum_{i=1}^m \frac{n}{p_i(\cdot)} \le C}$, then this assertion is valid for all values of $R > 0$ therefore it can be deduced that $\sum_{i=1}^m \frac{n}{p_i(\cdot)} = \frac{n}{p(\cdot)}$. Subsequently, consider $0 <\in< \min\left\{\frac{n\alpha_1(\cdot)}{p_1}, \ldots, \frac{n\alpha_m(\cdot)}{p_m}\right\}$ and proceed to define

$$g_{\in,K}(x) := \chi_{\{0\le|x|<1\}} + \sum_{j=1}^{k} \chi_{\{j\le|x|\le j+j^{-\in}\}}(x)$$

for any arbitrary integer $K \in \mathbb{N}$. For each $i = 1, \ldots, m$ define f_i as the function of $g_{\in,K}$. It can be noted that

$$\begin{aligned}\left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} &\ge \frac{1}{2}|B(0, K+K^{\in})|^{\frac{1}{p(\cdot)}-\frac{1}{\alpha(\cdot)}} \left|\left\{X \in B(a,r) : |f_i(x)| > \frac{1}{2}\right\}\right| \\ &\ge C(K+K^{-\in})^{\frac{d}{p(\cdot)}-\frac{d}{\alpha(\cdot)}}(K+K^{-\in})^{\frac{d}{p(\cdot)}-\frac{d}{\alpha(\cdot)}} \\ &= C(K+K^{-\in})^{\frac{d}{p(\cdot)}-\frac{d}{\alpha(\cdot)}}\end{aligned}$$

By leveraging Lemma 1 in conjunction with the Morrey-norm estimate pertaining to f_i, it is feasible to derive

$$\|f_i\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \le \|f_i\|_{{}_wMK^{\alpha_i(\cdot),\lambda}_{p_i(\cdot),q}} \le C(K+K^{-\in})^{\frac{d}{p(\cdot)}-\frac{d}{\alpha(\cdot)}}$$

For each i ranging from 1 to m . Given $\sum_{i=1}^m \frac{1}{p_i(\cdot)} = \frac{1}{p(\cdot)}$ and

$$\left\|\prod_{i=1}^m f_i\right\|_{{}_wMK^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \le m\prod_{i=1}^m \|f_i\|_{{}_wMK^{\alpha_i(\cdot),\lambda}_{p_i(\cdot),q}}$$

it follows that $(K+K^{-\in})^{-\frac{\in}{\alpha(\cdot)}+\sum_{i=1}^m \frac{\in}{\alpha_i(\cdot)}} \le C$.

As it valid $K \in \mathbb{N}$ for all, it can be inferred that $\sum_{i=1}^m \frac{1}{\alpha_i(\cdot)} \le \frac{1}{\alpha(\cdot)}$.

4. CONCLUSION

Herz-Morrey spaces are formally defined as spaces where functions satisfy specific conditions involving variable exponents, leading to the establishment of weak Herz-Morrey spaces with variable exponents. The conditions for the generalized Hölder's inequality with a variable exponent in Herz-Morrey spaces was investigated by providing mathematical proofs and analysis to support the results. The main Theorems established the equivalence of certain conditions for the generalized Hder's inequality in Herz-Morrey spaces with variable exponents, contributing to the understanding of Hölder's inequalities in such spaces.

5. ACKNOWLEDGMENT

The author would like to express gratitude to the Ministries of Education, Culture, Research, and Technology of the Republic of Indonesia for their support of this research through the PTM grant (No. 102/E5/PG.02.00.PL/2023).

REFERENCES

Almeida, A., J. Hasanov, and S. Samko (2008). Maximal and Potential Operators in Variable Exponent Morrey Spaces. *Georgian Mathematical Journal*, **15**(2); 195–208

Alshanti, W. G., I. M. Batiha, A. Alshanty, A. Zraiqat, and I. H. Jebril (2023). Perturbed Trapezoid Like Inequalities. *Science and Technology Indonesia*, **8**(2); 205–211

Aoyama, H. (2009). Lebesgue Spaces with Variable Exponent on a Probability Space. *Hiroshima Mathematical Journal*, **39**(2); 207–216

Chen, G. S., H. M. Srivastava, P. Wang, and W. Wei (2014). Some Further Generalizations of Hölder's Inequality and Related Results on Fractal Space. In *Abstract and Applied Analysis*, volume 2014. Hindawi, pages 802–832

Dung, K. H., C. D. L. Minh, and T. T. Nang (2023). Boundedness of Hardy-Cesàro Operators on Variable Exponent Morrey-Herz Spaces. *Filomat*, **37**(4); 1001–1016

Fan, X. (2010). Variable Exponent Morrey and Campanato Spaces. *Nonlinear Analysis: Theory, Methods & Applications*, **72**(11); 4148–4161

Gurka, P., P. Harjulehto, and A. Nekvinda (2007). Bessel Potential Spaces with Variable Exponent. *Mathematical Inequalities and Applications*, **10**(3); 661

Ifronika, M. Idris, A. A. Masta, and H. Gunawan (2017). Generalized Hölder's Inequality on Morrey Spaces. *arXiv*, **arXiv:1706.01659**; 1–12

Ifronika, A. A. Masta, M. Nur, and H. Gunawan (2018). Generalized Hölder's Inequality in Orlicz Spaces. *arXiv*, **arXiv:1809.00788**; 1–10

Izuki, M. (2010). Boundedness of Sublinear Operators on Herz Spaces with Variable Exponent and Application to Wavelet Characterization. *Analysis Mathematica*, **36**(1); 33–50

Masta, A. A., S. Fatimah, A. Arsisari, Y. Yunika Putra, and F. Apriani (2018). Sufficient and Necessary Conditions for Generalized Hölder's Inequality in p-Summable Sequence Spaces. *arXiv*, **arXiv:1812.06612**; 1–5

Masta, A. A., H. Gunawan, and W. Setya Budhi (2017). An Inclusion Property of Orlicz-Morrey Spaces. In *Journal of Physics: Conference Series*, volume 893. IOP Publishing, page 012015

Matkowski, J. (2009). A Converse of the Hölder Inequality Theorem. *Mathematical Inequalities & Applications*, **12**(1); 21–23

Mizuta, Y. (2016). Duality of Herz-Morrey Spaces of Variable Exponent. *Filomat*, **30**(7); 1891–1898

Mizuta, Y. and T. Ohno (2015). Herz–Morrey Spaces of Variable Exponent, Riesz Potential Operator and Duality. *Complex Variables and Elliptic Equations*, **60**(2); 211–240

Nakai, E. and Y. Sawano (2012). Hardy Spaces with Variable Exponents and Generalized Campanato Spaces. *Journal of Functional Analysis*, **262**(9); 3665–3748

Nogayama, T. (2019). Boundedness of Commutators of Fractional Integral Operators on Mixed Morrey Spaces. *Integral Transforms and Special Functions*, **30**(10); 790–816

Orlicz, W. (1931). Über Konjugierte Exponentenfolgen. *Studia Mathematica*, **3**(1); 200–211

Shanzhen, L. and X. Lifang (2005). Boundedness of Rough Singular Integral Operatorson the Homogeneous Morrey-Herz Spaces. *Hokkaido Mathematical Journal*, **34**(2); 299–314

Sultan, B., F. M. Azmi, M. Sultan, T. Mahmood, N. Mlaiki, and N. Souayah (2022). Boundedness of Fractional Integrals on Grand Weighted Herz–Morrey Spaces with Variable Exponent. *Fractal and Fractional*, **6**(11); 660

Vasyunin, V. (2004). The Sharp Constant in the Reverse Hölder Inequality for Muckenhoupt Weights. *St. Petersburg Mathematical Journal*, **15**(1); 49–79

Wajih, A. and H. Gunawan (2020). On Generalized Hölder's Inequality in Weak Morrey Spaces. *Journal of the Indonesian Mathematical Society*, **26**(01); 165–169

Wang, H. and F. Liao (2020). Boundedness of Singular Integral Operators on Herz-Morrey Spaces with Variable Exponent. *Chinese Annals of Mathematics, Series B*, **41**(1); 99–116

Wang, H. and Y. Wu (2016). Anisotropic Herz-Morrey Spaces with Variable Exponents. *Khayyam Journal of Mathematics*, **2**(2); 177–187

Xu, J. and X. Yang (2015). Herz-Morrey-Hardy Spaces with Variable Exponents and Their Applications. *Journal of Function Spaces*, **2015**(Article ID 160635); 1–19

