Science and Technology Indonesia
e-ISSN:2580-4391 *p*-ISSN:2580-4405
Vol. 11, No. 3, July 2026

Research Paper

The Locating-Chromatic Number of Disjoint Union of Cycles

Des Welyyanti[1*], Muhammad Rafif Fajri[1,2], Latifa Azhar Abel[1], Lyra Yulianti[1], Aisyah Nurinsani[1], Dony Permana[3]

[1]*Department of Mathematics and Data Sciences, Faculty of Mathematics and Natural Sciences, Universitas Andalas, Padang, West Sumatra, 25163, Indonesia*
[2]*Doctoral Program of Mathematics, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, West Java, 40132, Indonesia*
[3]*Department of Statistics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Padang, Padang, West Sumatra, 25171, Indonesia*
*__Corresponding author__: wely@sci.unand.ac.id

Abstract
Chartrand et al. introduced the idea of the locating-chromatic number of connected graphs in 2002. Let c be a coloring of a graph H with k-colors. Let S_i be the set of all vertices that get color i and let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be the partition of $V(H)$ induced by c. The color code $c_\Pi(v) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$ of a vertex $v \in H$, where $d(v, S_i) = \min\{d(v, x) \mid x \in S_i\}$ and $d(v, S_i) < \infty$ for all $i \in [1, k]$. The locating k-coloring of H is denoted by c if all vertices in H have unique distinct color codes. Welyyanti et al. in 2014 expanded on this idea so that it also applies to unconnected graphs. In this work, for $n \geq 3$ and $m \geq 2$, we calculate the locating-chromatic number of the disjoint union of cycles, represented by mC_n.

Keywords
Color Code, Locating Chromatic Number, Cycle Graph

Received: 20 January 2026, Accepted: 18 May 2026
https://doi.org/10.26554/sti.2026.11.3.1046-1053

1. INTRODUCTION

The locating-chromatic number of connected graphs was introduced by Chartrand et al. (2002). Many notable findings have been produced. For example, the locating-chromatic number of tree, cycles and path graphs were calculated by them. Asmiati and Baskoro (2012) described all graphs with cycles on n vertices that have locating-chromatic number 3 finalized the characterization the tree graph with $X_L = 3$.

Asmiati et al. (2018) have conducted extensive research on the locating chromatic number of barbell graphs, such as discussed the certain barbell graphs (Asmiati et al., 2018), the barbell shadow path graphs (Asmiati et al., 2021) and subdivision of barbell graphs containing generalizied Petersen graph. Some authors also studied the locating-chromatic number for path graph and cycle graph. Damayanti et al. (2021) determined some modified path with cycle, Ghanem et al. (2019) disccussed the power of paths and cycles, Hamzah et al. (2024) research the corona operation of path P_n and cycle C_m with $m = 3, 4$. Sakri and Abbas (2024) researchs the locating chromatic number for a generalized petersen graphs with small order and generalized petersen graphs $P(N, 2)$ Sakri and Slimi (2025).

Certain authors have also investigated the locating-chromatic number for graphs created through various graph operations. Chartrand et al. (2002) were the first to investigate the locating-chromatic number for trees by demonstrating this number for double stars and paths. Additionally, Chartrand et al. (2003) demonstrated that for every integer $k \in [3, n]$ and $k \neq n - 1$, a tree with n vertices with $X_L = k$. Asmiati et al. (2012) identified the firecrackers and the amalgamation of stars (Asmiati et al., 2011). Abel et al. (2025) investigated the cyclic chain graph. Then Welyyanti et al. (2025) studies about $Chain(A, 4, n)$, small circulant graph (Welyyanti et al., 2026a) and sunflower graph (Welyyanti et al., 2026b). There are research on the Petersen graph (Arawan and Istiani, 2024), the origami graph (Irawan et al., 2021).

For some corona graph, Hamzah et al. (2025) determine the locating chromatic number for corona of path and cycle (Hamzah et al., 2024). Then, in 2025 They continued the research with application with python (Hamzah et al., 2025). Another graph, research in the locating chromatic number are jellyfish graphs (Arfin, 2025), m-shadow of a connected graph (Sudarsana et al., 2022), (k, n)-split cycle graph and its barbell operation (Asmiati et al., 2025), all unicyclic graphs of order n (Baskoro and Arfin, 2021), upper bounds of the locating chromatic numbers of shadow cycle graphs (Asmiati et al., 2024), and the edge-amalgamation of trees (Assiyatun et al., 2020).

In this research, we find the locating-chromatic number for the disjoint union of cycles. Let c be a k-coloring of a disconnected graph H, $c : V(H) \rightarrow \{1, 2, \ldots, k\}$ be a vertex coloring such that two neighboring vertices are colored in different colors. If S_i is the set of all vertices that are allocated the same color receive color i for $1 \leq i \leq k$, i a color class for a vertices graph H, and let $\Pi = \{S_1, S_2, \dots, S_k\}$, where S_i be the partition of $V(H)$ induced by c.

The color code of a vertex v in H is defined as a k-vector: $c_\Pi(v) = (d(v, S_1), d(v, S_2), \dots, d(v, S_k))$ where $d(v, S_k) = \min\{d(v, x) \mid x \in S_i\}$ is the distance between a vertex v in H and $d(v, S_k) < \infty$ for all $i \in [1, k]$. The locating k-coloring of H is denoted by c if all vertices in H have unique distinct color code. The symbol $\chi'_L(H)$ is the value of the locating-chromatic number of a disconnected H graph.

The following theorems and lemma are useful in determining the locating-chromatic number of disjoint union of m cycles for $m \geq 2$.

Theorem 1. *Let $H = \cup_{i=1}^{m} G_i$ For $1 \leq i \leq m$, and G_i is a disconnected graph. If $\chi'_L(H) < \infty$, then H does not contain any two components G_i and G_j such that $\chi_L(G_i) = G_i$, $\chi_L(G_j) = G_j$, $|G_j| \leq q \leq \chi'_L(H) \leq r \leq |G_i|$, where $r = \min\{|V(G_i)| : i \in [1, m]\}$, $q = \max\{\chi'_L(G_i) : i \in [1, m]\}$ and $|G_j| \neq |G_i|$.*

Lemma 1. *Let Π be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then u and v belong to different elements of Π.*

Theorem 2. *Let C_n be a cycle on n vertices for $n \geq 3$. Then*

$$X'_L(C_n) = \begin{cases} 3, & \text{for odd } n \\ 4, & \text{for even } n \end{cases}$$

2. EXPERIMENTAL SECTION

(a) Based on Theorem 2, establish that the lower bound of the locating chromatic number for the disjoint union of cycles graph is 3 or 4, due to the nature of the disjoint union of cycles graph. If the lower bound fails to satisfy the locating chromatic number criteria, then adding vertex colors the requirements for the locating chromatic number.

(b) Established the upper bound of the locating chromatic number for the disjoint union of cycles graph, which will be demonstrated by showing that the color code of every vertex in the disjoint union of cycles graph must differ.

3. RESULTS AND DISCUSSIONS

Let integers $m \geq 1$ and $n \geq 3$, we write

$$mC_n = \bigcup_{i=1}^{m} C_n^{(i)},$$

The disjoint union of m copies of the cycle C_n, where the components $C_n^{(1)}, C_n^{(2)}, \dots, C_n^{(m)}$ are pairwise vertex-disjoint. In the following theorems, it is proved that the locating chromatic number of disjoint union of odd cycles mC_n is 3 if and only if $m = 1$.

Theorem 3. *Let C_n be a cycle on n vertices with $n \geq 3$ be odd and m is copies of the cycle C_n, denote mC_n. Let $m \geq 1$ than $X'_L(mC_n) = 3$ if only if $m = 1$.*

Proof. If $m = 1$, then $mC_n = C_n$. Since n is odd, by Theorem 2 we have $X'_L(C_n) = 3$. Assume $X'_L(mC_n) = 3$. Then mC_n admits a locating 3-coloring. In particular, for each component $C_n^{(1)}$ we obtain a locating coloring of $C_n^{(i)}$ using at most 3 colors, so $C_n^{(1)} \leq 3$. Because n is odd, Theorem 2 gives $X'_L(C_n^{(i)}) = X'_L(C_n) = 3$ for every i. If $m \geq 2$, then mC_n contains two distinct components $2C_n^{(1)}$ and $2C_n^{(2)}$ with $X'_L(2C_n^{(1)}) = X'_L(2C_n^{(2)}) = X'_L(2C_n^{(i)}) = 3$, which contradicts Theorem 1 (with $k = 3$). Therefore $m = 1$.

Theorem 4. *Let $n \geq 4$. If $X'_L(mC_n) < \infty$, then $m \leq \frac{(n-1)!}{2\lfloor \frac{n-1}{2} \rfloor}$.*

Proof. Let $G = mC_n$ where $n \geq 4$ and $V(G) = \{v_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

Let $G = mC_n = \bigcup_{i=1}^{m} C_n^{(i)}$ and suppose G admits a locating coloring. In particular, G admits a locating n-coloring; let $c : V(G) \to [n]$ be such a coloring and let $\Pi = \{S_1, S_2, \dots, S_n\}$ be the partition induced by c, where $S_j = \{v \in V(G) : c(v) = j\}$. For each component $C_n^{(i)}$, since we use n colors on a cycle of order n, each color appears exactly once in $C_n^{(1)}$.

Fix the unique vertex $x_i \in V(C_n^{(i)})$ with color 1, i.e., $x_i \in S_1$. Write the color code of x_i as

$$\begin{aligned} c_\Pi(x_i) &= (d(x_i, S_1), d(x_i, S_2), \dots, d(x_i, S_n)) \\ &= (0, a_{i,2}, a_{i,3}, \dots, a_{i,n}), \end{aligned}$$

where $a_{i,j} = d(x_i, S_j)$ is the distance in $C_n^{(i)}$ from x_i to the unique vertex of color j in that component. Because all vertices $x_i, \dots, x_n$ have the same color 1, the locating property forces their color codes to be pairwise distinct:

$$c_\Pi(x_i) \neq c_\Pi(x_{i'}), \text{ for all } i \neq i'.$$

Hence, m is at most the number of distinct n-tuples $(0, a_2, a_3, \dots, a_n)$ that can occur for a color one vertex in a single cycle C_n.

Now count these possibilities using the structure of distances in C_n:

Case 1: n odd

We have $diam(C_n) = \frac{n-1}{2}$ and for each $t \in \{1, 2, \dots, \frac{n-1}{2}\}$. There are exactly two vertices at distance t from x_i in $C_n^{(i)}$. Thus, among the colors $\{2, 3, \dots, n\}$, exactly two colors must be placed at distance 1, two colors at distance 2, and so on. Therefore, the number of possible assignments of the distances $(a_{i,2}, a_{i,3}, \dots, a_{i,n})$, is at most at most such as Equation (1).

$$m \leq \binom{n-1}{2}\binom{n-3}{2}\binom{n-5}{2}\cdots\binom{4}{2}\binom{2}{2} = \frac{(n-1)!}{2^{\frac{n-1}{2}}} \tag{1}$$

Case 2: n even

We have $diam(C_n) = \frac{n}{2}$ and for each $t \in \{1, 2, \dots, \frac{n}{2} - 1\}$. There are exactly two vertices at distance t from x_i, and there is exactly one vertex at distance $\frac{n}{2}$. Hence, we choose two colors to

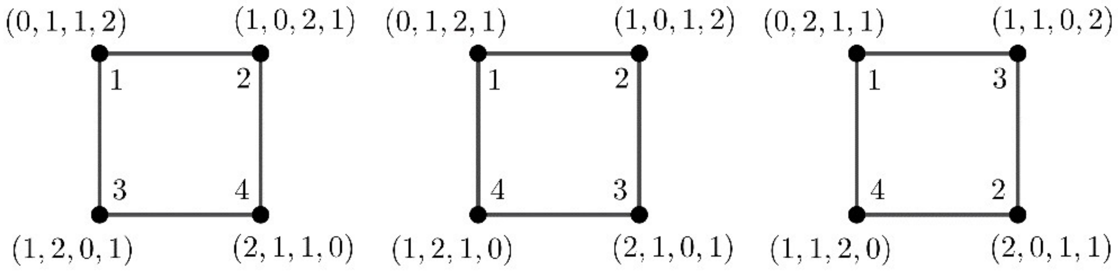

Figure 1. Color Code and Locating 4-Coloring of $3C_4$

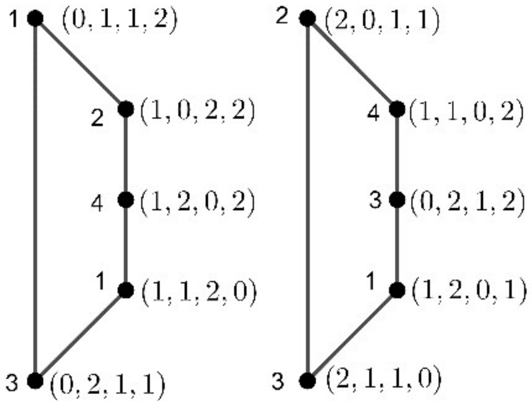

Figure 2. Locating 4-Coloring and Color Code of $2C_5$

occupy distance 1, two colors for distance 2, ..., and two colors for distance $\frac{n}{2} - 1$ and the remaining single color is forced to be at distance $\frac{n}{2}$. Thus, the value of m such as Equation (2).

$$m \le \binom{n-1}{2}\binom{n-3}{2}\binom{n-5}{2}\cdots\binom{5}{2}\binom{3}{2} = \frac{(n-1)!}{2^{\frac{n-2}{2}}} \quad (2)$$

In both cases, (2) and (3) can be written as $m \le \frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor}}$.

Corollary 1. *Let $n \ge 4$. Define*

$$k := \min\{t \ge 1 : X'_L(tC_n) = n\}.$$

If $X'_L(mC_n) < n$, then $m < k$. Moreover,

$$X'_L(mC_n) = \begin{cases} n, & k \le m \le \frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor}} \\ \infty, & m > \frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor}} \end{cases} \quad (3)$$

Proof. Let $G = mC_n$.

First, by the definition of k, for every $m \ge k$ we have $X'_L(mC_n) \ge n$, while clearly $X'_L(mC_n) < n$ whenever an n-locating coloring exists. Hence, if $X'_L(mC_n) < n$ then necessarily $m < k$.

Now assume $m \ge k$. By Theorem 4 if $X'_L(mC_n) < \infty$, then $m \le \frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor}}$. Therefore, for every $k \le m \le \frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor}}$ we have if $X'_L(mC_n) = n$ (because if $X'_L(mC_n) \ge n$ by the definition of k, and $X'_L(mC_n) \le n$ since the same n colors are used).

Finally, if $m > \frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor}}$, then Theorem 4 implies that no locating coloring exists; hence $X'_L(mC_n) = \infty$. Then the Equation (3) is proven.

Theorem 5. *Let mC_4 is a cycle graph of m with C_4. Then,*

$$X'_L(mC_4) = \begin{cases} 4, & 1 \le m \le 3 \\ \infty, & \textit{otherwise.} \end{cases} \quad (4)$$

Proof. By Theorem 1 and Theorem 2, we have $X_L(C_4) = 4$. Therefore, $X'_L(mC_4) \ge 4$.

Case 1. By Corollary 1, we have $m \le \frac{(3-1)!}{2^{\lfloor \frac{3-1}{2} \rfloor}} = 3$. Then, $X'_L(mC_4) \le 4$. Thus, for $1 \le m \le 3$, we have $X'_L(mC_4) = 4$. The locating-coloring and color code of $3C_4$ can be obtained from Figure 1.

Case 2. By Lemma 1, given that $X'_L(mC_4) \ge 4$ and every vertex has to be connected to two other partition classes, it follows that $X'_L(mC_4) = \infty$ for all $m \ge 4$.

Based on Case 1 and Case 2, then Equation (4) is proven.

Theorem 6. *Let mC_5 is a cycle graph of m with order 5. Then,*

$$X'_L(mC_5) = \begin{cases} 3, & m = 1, \\ 4, & m = 2, \\ 5, & 3 \le m \le 6 \\ \infty, & \textit{otherwise.} \end{cases} \quad (5)$$

Proof. Let $G = mC_5$ and define the vertex set of G as $V(G) = \{v_{i,j} \mid 1 \le i \le m,\ 1 \le j \le 5\}$. From Theorem 3, because $X'_L(C_5) = 3$, then $X'_L(G) \ge 3$.

Case 1. If $m = 1$, then from Theorem 3, we get $X'_L(G) = 3$.

Case 2. We show that $m > 1$ implies $X'_L(G) \ge 4$ i.e., no locating 3-coloring exists. Assume to the contrary that G admits a locating 3-coloring c. Let $\Pi = \{S_1, S_2, S_3\}$ be the color classes induced by c, where $S_j = \{v \in V(G) : c(v) = j\}$. Fix any component $C_5^{(i)}$ of G and a vertex $v \in V(C_5^{(i)})$. Since C_5 is an odd cycle, each component $C_5^{(i)}$ must use all three colors (otherwise it would be a proper 2-coloring of C_5 impossible). Hence, for every $j \in \{1, 2, 3\}$, the distance $d(v, S_j)$ is finite

Table 1. Color Code of $8C_6$

Color Code for 1	Representation in Figure	Same Color Code in Vertex
(0,1,1,2,2)	Figure 5. (6)	-
(0,1,2,1,2)	Figure 5. (4)	-
(0,1,2,2,1)	Figure 5. (2)	-
(0,2,1,1,2)	Figure 5. (2)	-
(0,2,1,2,1)	Figure 5. (4)	-
(0,2,2,1,1)	Figure 5. (6)	-
(0,1,1,2,3)	Figure 5. (1)	-
(0,1,1,3,2)	Figure 5. (7)	-
(0,1,2,1,3)	-	$v_{3,2}$ or $v_{3,4}$
(0,1,2,3,1)	Figure 5. (3)	-
(0,1,3,1,2)	-	$v_{7,4}$ or $v_{7,6}$
(0,1,3,2,1)	Figure 5. (2)	-
(0,2,1,1,3)	-	$v_{5,2}$ or $v_{5,4}$
(0,2,1,3,1)	Figure 5. (5)	-
(0,2,3,1,1)	Figure 5. (5)	-
(0,3,1,1,2)	Figure 5. (7)	-
(0,3,1,2,1)	-	$v_{1,4}$ or $v_{1,6}$
(0,3,2,1,1)	Figure 5. (3)	-
(0,1,2,2,3)	-	$v_{2,2}$
(0,1,2,3,2)	-	$v_{4,3}$
(0,1,3,2,2)	-	$v_{6,3}$
(0,2,1,2,3)	-	$v_{4,2}$
(0,2,1,3,2)	Figure 5. (8)	-
(0,2,2,1,3)	-	$v_{6,2}$
(0,2,2,3,1)	-	$v_{6,6}$
(0,2,3,1,2)	-	$v_{2,5}$
(0,2,3,2,1)	-	$v_{4,6}$
(0,3,1,2,2)	-	$v_{6,5}$
(0,3,2,1,2)	-	$v_{4,5}$
(0,3,2,2,1)	-	$v_{2,6}$

and is computed entirely within the component $C_5^{(i)}$ (vertices in other components are infinite distance and do not affect the minimum).

Because $diam(C_5) = 2$, every entry of the color code $c \in V(G)$ belongs to $\{0, 1, 2\}$. Moreover, if $v \in S_1$, then $d(v, S_1) = 0$ and the two neighbors of v must lie in $S_2 \cup S_3$. There are only three possibilities:

Case 1: the two neighbors have colors 2 and 3, hence $c_\Pi(v) = (0, 1, 1)$;

Case 2: both neighbors have color 2, hence the nearest vertex of color 3 is at distance 2, so $c_\Pi(v) = (0, 1, 2)$;

Case 3: both neighbors have color 3, hence $c_\Pi(v) = (0, 2, 1)$.

By symmetry, if $v \in S_2$ then $c_\Pi(v) \in \{(1, 0, 1), (1, 0, 2), (2, 0, 1)\}$, and if $v \in S_3$ then $c_\Pi(v) \in \{(1, 1, 0), (2, 1, 0), (1, 2, 0)\}$. Therefore, under any proper 3-coloring of G, every vertex has a color code belonging to the following set of at most nine vectors: $(0, 1, 1)$, $(0, 1, 2)$, $(0, 2, 1)$, $(1, 0, 1)$, $(2, 0, 1)$, $(1, 0, 2)$, $(1, 1, 0)$, $(2, 1, 0)$ and $(1, 2, 0)$. Hence G can have at most 9 distinct color codes under a 3-coloring. But $|V(G)| = 5m \geq 10$ for $m > 1$, so by the pigeonhole principle two distinct vertices of G must share the same color code, contradicting that c is locating. Thus no locating 3-coloring exists, and $X'_L(G) \geq 4$ for $m > 1$. Since $diam(C_5) = 2$, then for any 3-coloring of C_5, there are nine different possibilities of color code under such partition: $(0, 1, 1)$, $(0, 1, 2)$, $(0, 2, 1)$, $(1, 0, 1)$, $(2, 0, 1)$, $(1, 0, 2)$, $(1, 1, 0)$, $(2, 1, 0)$ and $(1, 2, 0)$. So, we have $X'_L(mC_5) \geq 4$ for $m > 1$.

Let $m = 2$, then to show that $X'_L(G) \leq 4$, define a partition $\Pi = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{v_{1,1}, v_{1,4}, v_{2,4}\}$, $S_2 = \{v_{1,2}, v_{2,1}\}$, $S_3 = \{v_{1,5}, v_{2,3}, v_{2,5}\}$, and $S_4 = \{v_{1,3}, v_{2,2}\}$. Therefore, Π constitutes a resolving partition of $V(G)$ where $G = 2C_5$. Therefore, $X'_L(G) = 4$ for $m = 2$. The locating 4-coloring and color code of $2C_5$ can be taken from Figure 2.

Case 3. First, we will show if $m = 3$ then $X'_L(G) \geq 5$. Without loss of generality, consider the vertex with color one. The number of color codes $c_\Pi(v(G)) = (d(v, S_1)) = 1$. If with 4-coloring, $c_\Pi(v(G)) = \{(0, 1, 1, 2), (0, 1, 2, 1), (0, 2, 1, 1), (0, 1, 2, 2), (0, 2, 1, 2), (0, 2, 2, 1)\}$. The color

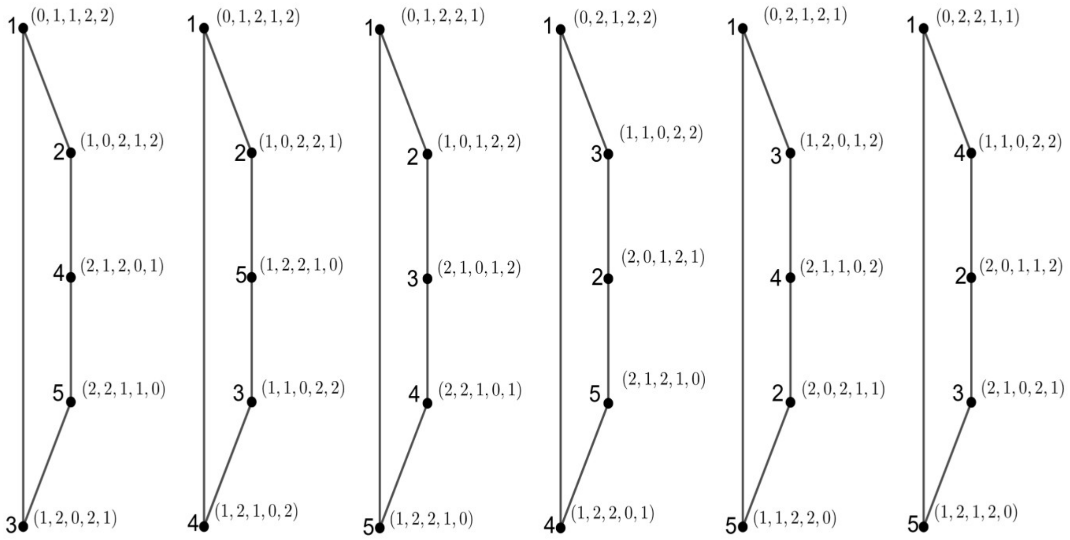

Figure 3. Locating 5-Coloring and Color Code of $6C_5$

codes $(0, 1, 1, 2)$, $(0, 2, 1, 1)$, and $(0, 2, 1, 2)$ are already used in Case 2, so we pay attention only to the other three color codes. For color code $(0, 1, 2, 1)$ and $(0, 2, 1, 1)$ have same color code for vertex $v_{1,4}$ or $v_{2,5}$ in Case 2. For color code $(0, 1, 2, 2)$ have same color code for vertex $v_{1,2}$ in Case 2. Therefore, for $m = 3$, $X'_L(G) \geq 5$.

Next, we will show that if $m = 6$ then $X'_L(G) \leq 5$. By Corollary 1, we have $X'_L(G) \leq \frac{(5-1)!}{2^{\lfloor \frac{5-1}{2} \rfloor}} = 6$. Therefore, for $3 \leq m \leq 6$, we have $X'_L(G) = 5$. The locating-coloring and color code of $6C_5$ can be taken from Figure 3.

Case 4. By Corollary 1 we have $m > \frac{(5-1)!}{2^{\lfloor \frac{5-1}{2} \rfloor}} = 6$, thus, for $m \geq 7$, $X'_L(G) = \infty$.

Based on Case 1 until Case 4, then Equation (5) is proven.

Theorem 7. *Let $1 \leq m \leq 7$. For each $i \in [1, m]$, let C_{n_i} be a cycle of even order $n_i \geq 6$. Then*

$$X'_L\left(\bigcup_{i=1}^{m} C_{n_i}\right) = 4.$$

Proof. For $n_i \geq 6$, let $G = \cup_{i=1}^{m} C_{n_i}$ and $V(C_{n_i}) = \{v_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n_i\}$. By Theorem 1 and Theorem 2, since $X_L(C_n) = 4$ for $n \geq 4$ and n even, then $X'_L(G) \geq 4$.

To show $X'_L(G) \leq 4$, define a 4-coloring $c : V(G) \to \{1, 2, 3, 4\}$ as follows. Let

$$\langle t \rangle_4 := 1 + ((t-1) \bmod 4) \in \{1, 2, 3, 4\}.$$

For $1 \leq i \leq m$ and $1 \leq j \leq n_i$, set

$$c(v_{i,j}) = \begin{cases} 4, & (i,j) \in \{(1,1), (2,2), (3,3)\}, \\ 3, & (i,j) \in \{(1,2), (2,1), (3,1)\}, \\ 2, & (i,j) = (3,2), \\ 2, & 1 \leq i \leq 3 \text{ and } j \geq 3 \text{ odd}, \\ 1, & 1 \leq i \leq 3 \text{ and } j \geq 4 \text{ even}, \\ \langle i-2 \rangle_4, & 4 \leq i \leq m \text{ and } j = 2, \\ \langle i-1 \rangle_4, & 4 \leq i \leq m \text{ and } j = 4, \\ \langle i-3 \rangle_4, & 4 \leq i \leq m \text{ and } j \text{ odd}, \\ \langle t \rangle_4 & 4 \leq i \leq m \text{ and } j \geq 4 \text{ even}. \end{cases}$$

Let $\Pi = \{S_1, S_2, S_3, S_n\}$ be the partition induced by c. It is straightforward to check that c is a proper coloring on each component C_{n_i} (since adjacent indices have opposite parity, and the special assignments at $j = 2, 4$ are different from their neighbors). Moreover, the resulting color codes are pairwise distinct (see Figure 4 for the explicit color codes), hence Π is locating. Therefore, $X'_L(\cup_{i=1}^{m} G) \leq 4$. Combining with the lower bound gives $X'_L(\cup_{i=1}^{m} G) = 4$.

Theorem 8. *Let C_6 represent a cycle of order 6. Then,*

$$X'_L(mC_6) = \begin{cases} 4, & 1 \leq m \leq 7, \\ 5, & m = 8, \\ 6, & 9 \leq m \leq 30, \\ \infty, & \textit{otherwise}. \end{cases} \tag{6}$$

Proof. Let $G = mC_6$ and $V(G) = \{v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 6\}$. By Theorem 1 and Theorem 2, we have $X_L(C_6) = 4$. Therefore, $X'_L(G) \geq 4$.

Case 1. If $1 \leq m \leq 7$. Using Theorem 7, we get, $X'_L(G) = 4$.

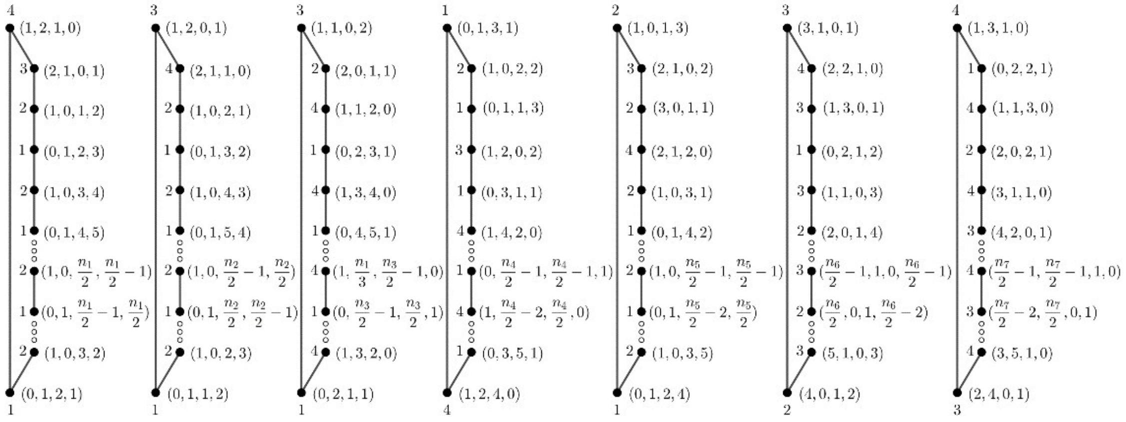

Figure 4. The Locating 4-Coloring and Color Code of $7C_{ni}$

Case 2. First, We show that $X'_L(G) \geq 5$ for $m = 8$. Assume to the contrary that G admits a resolving 4-coloring $c : V(G) \to \{1, 2, 3, 4\}$, and let $\Pi = \{S_1, S_2, S_3, S_n\}$ be the induced partition. Consider a single component $C_6^{(i)}$ and a vertex $v \in C_n^{(i)}$ with $c(v) = 1$. Since $diam(C_6) = 3$ the color code of v has the form

$$c_\Pi(v) = (0, a_2, a_3, a_4), \quad a_j \in \{1, 2, 3\}$$

Because c is resolving on $C_6^{(i)}$, all three colors $\{2, 3, 4\}$ must appear in $C_6^{(i)}$. Moreover, at least two of them must appear in the closed neighborhood $N(v)$, otherwise two vertices symmetric with respect to v would have identical distances to all color classes, contradicting resolvability. Hence, for a color-1 vertex v, exactly two of the distances a_2, a_3, a_4 equal 1, while the remaining one equals 2 or 3. Up to permutation of colors $\{2, 3, 4\}$, this yields precisely the following six possible color codes:

$$(0, 1, 1, 2), (0, 1, 2, 1), (0, 2, 1, 1),$$
$$(0, 1, 1, 3), (0, 1, 3, 1), \text{ and } (0, 3, 1, 1).$$

Furthermore, by the cyclic symmetry of C_6, every resolving 4-coloring of $C_6^{(i)}$ must realize all six patterns; if one such pattern were missing, then two vertices at opposite positions in the cycle would share the same distances to all four color classes, again contradicting resolvability. Consequently, in $8C_6$ there are at most 6 distinct color codes among vertices colored 1. Since there are 8 components, by the pigeonhole principle two vertices of color 1 in different components must share the same color code, contradicting that c is resolving on G. Therefore, no resolving 4-coloring exists for $m = 8$ and hence $X'_L(G) \geq 5$.

Next, we will show that $X'_L(G) \leq 5$ for $m = 8$. Define a partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $V(G)$, than:

$$S_1 = \{v_{i,1}, v_{1,3}, v_{2,4}, v_{3,5}, v_{4,4}, v_{5,5}, v_{6,4}, v_{7,3} \mid i \in [1, 8]\},$$
$$S_2 = \{v_{1,2}, v_{2,2}, v_{3,2}, v_{4,3}, v_{5,3}, v_{6,3}, v_{7,6}, v_{8,5}\},$$
$$S_3 = \{v_{1,4}, v_{2,3}, v_{3,3}, v_{4,2}, v_{5,2}, v_{6,5}, v_{7,2}, v_{8,2}, v_{8,6}\},$$
$$S_4 = \{v_{1,5}, v_{2,5}, v_{3,4}, v_{4,5}, v_{5,4}, v_{6,2}, v_{7,4}, v_{8,4}\},$$
$$S_5 = \{v_{1,6}, v_{2,6}, v_{3,6}, v_{4,6}, v_{5,6}, v_{6,6}, v_{7,5}, v_{8,3}\}.$$

Every vertex in $8C_6$ have different color codes. Thus, $X'_L(G) \leq 5$. Therefore, $X'_L(G) = 5$ for $m = 8$. The locating 5-coloring and color code of $8C_6$ can be taken from Figure 5.

Case 3. First, we will show that for $m = 9$, then, $X'_L(G) \geq 6$. Without loss of generality, consider the vertex with color 1. The number color codes $C_\Pi(V(G)) = 1$ with 5-coloring is 30, shown in Table 1. For example, we can use the color code $(0, 1, 1, 2, 2)$ and it is shown in Figure 5 (2). While, the color code $(0, 1, 2, 1, 3)$ already belongs to vertices $v_{3,2}$ and $v_{3,4}$, so this color code cannot be used anymore. By Table 1. we know that for $m = 9$, then $X'_L(G) \geq 6$. Next, we will show that for $m = 30$, $X'_L(G) \leq 6$. By Corollary 1, we have that $m \leq \frac{(6-1)!}{2^{\lfloor \frac{6-1}{2} \rfloor}} = 30$. Therefore, for $9 \leq m \leq 30$, $X'_L(G) = 6$.

Based on Table 1, the color code for $(0, 1, 1, 2, 2)$, no vertex same color code in vertex, but the color code for $(0, 3, 2, 2, 1)$ have the same color code in vertex $v_{2,6}$. Because there are several color codes that have more than 1 vertex, so, the $8C_6$ cannot use 4-coloring. Case 4. From Corollary 1, we have that $X'_L(G) = \infty$ for all $m \geq 31$. Based on Case 1 until Case 4, then Equation (6) is proven.

Theorem 9. *Let $n \geq 6$ and let $G = mC_n$. Put $f = \lfloor \frac{n-1}{2} \rfloor$. If*

$$\left\lfloor \frac{(n-1)^2(n-2)!}{n(2^f)} \right\rfloor + 1 \leq m \leq \frac{(n-1)!}{2^f},$$

then $X'_L(mC_n) = n$.

Proof. Assume a component C_n is properly colored with exactly $n - 1$ colors, and fix a vertex v of color 1. Let $\Pi =$

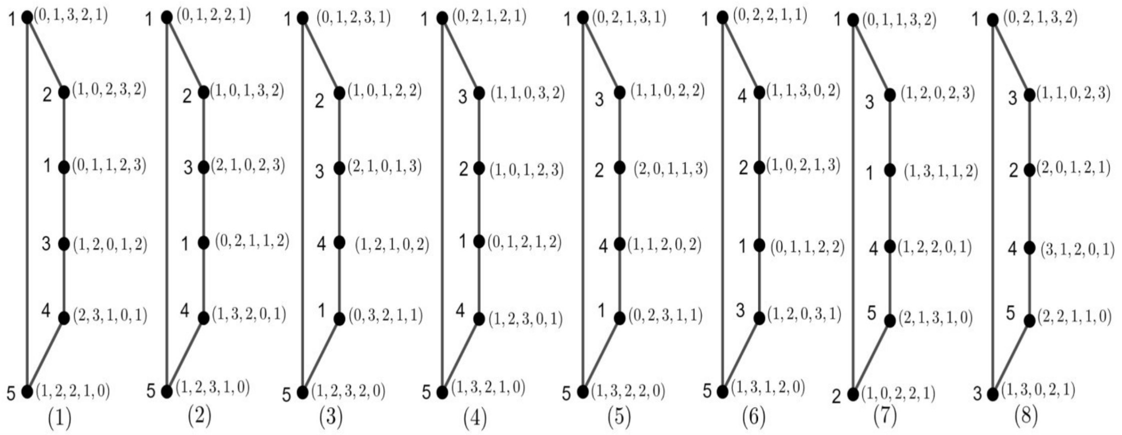

Figure 5. Locating 4-Coloring and Color Code of $8C_6$

$\{S_1, S_2, \ldots, S_{n-1}\}$ be the color classes. We count the maximum number k of distinct color codes $C_\Pi(v)$ that can occur.

Case 1: n odd. Let $D = \frac{n-1}{2} = diam(C_n)$. For each distance layer $t \in \{1, 2, \ldots, D\}$ there are exactly two vertices at distance t from v. Excluding color 1, there are $n-2$ colors to be realized as first occurrences among $n-1$ vertex positions, hence exactly one position is a repeat and therefore exactly one distance layer contributes only one new color. Choose that layer D choices, choose the single new color there ($\binom{n-2}{1}$ choices), then distribute the remaining $n-3$ colors into $D-1$ layers, two per layer:

$$k = D\binom{n-2}{1}\binom{n-3}{2}\binom{n-5}{2}\cdots\binom{2}{2} = \frac{(n-1)!}{2^D} = \frac{(n-1)!}{2^f}$$

Case 2: n even. Let $D = \frac{n}{2} = diam(C_n)$. Now layers $1, 2, \ldots, D-1$ have two vertices each, while layer D has one vertex. Again we need $n-2$ new colors but have $n-1$ positions, so one position is a repeat. There are two sub cases:

(A). The unique vertex at distance D is a repeat (so no color has nearest distance D). Then all $n-2$ colors appear first among the $2(D-1) = n-2$ positions in layers $1, 2, \ldots, D-1$, two per layer:

$$k_A = \binom{n-2}{1}\binom{n-4}{2}\cdots\binom{2}{2} = \frac{(n-2)!}{2^{D-1}}$$

(B). The vertex at distance D introduces a new color (choose it in $n-2$ ways), so the repeat must happen in one of the $D-1$ double-layers (choose that layer in $D-1$ ways). After fixing these choices, we choose one new color for the "single" layer ($\binom{n-3}{1}$ ways) and pair the remaining colors over the remaining layers:

$$\begin{aligned} k_B &= (D-1)(n-2)\binom{n-3}{1}\binom{n-4}{2}\binom{n-6}{2}\cdots\binom{2}{2} \\ &= (D-1)(n-2)\frac{(n-3)!}{2^{D-2}} \\ &= \frac{(n-2)(n-2)!}{2^{D-1}} \end{aligned}$$

Hence,

$$\begin{aligned} k = k_A + k_B &= \frac{(n-2)!}{2^{D-1}} + \frac{(n-2)(n-2)!}{2^{D-1}} \\ &= \frac{(n-1)(n-2)!}{2^{D-1}} = \frac{(n-1)!}{2^{D-1}} = \frac{(n-1)!}{2^f} \end{aligned}$$

Thus, for all $n \geq 6$,

$$k = \frac{(n-1)!}{2^f}$$

Suppose, for contradiction, that $G = mC_n$ has a locating $(n-1)$-coloring. With $n-1$ colors and $|V(G)| = mn$ vertices, some color class has size at least $\lceil \frac{mn}{n-1} \rceil$. All vertices in a single color class must have pairwise distinct color codes, so $\lceil \frac{mn}{n-1} \rceil \leq k$. Therefore, if $m > \frac{(n-1)k}{n}$, no locating $(n-1)$-coloring exists, i.e. $X'_L(mC_n) \geq n$. Using $k = \frac{(n-1)!}{2^f}$ gives

$$\frac{(n-1)k}{n} = \frac{(n-1)^2(n-2)!}{n(2^f)},$$

So a sufficient condition is

$$m \geq \left\lfloor \frac{(n-1)^2(n-2)!}{n(2^f)} \right\rfloor + 1$$

For $m \leq \frac{(n-1)!}{2^f}$, by Corollary 1 we have $X'_L(mC_n) \leq n$. Therefore, we obtain $X'_L(mC_n) = n$, if

$$\left\lfloor \frac{(n-1)^2(n-2)!}{n(2^f)} \right\rfloor + 1 \leq m \leq \frac{(n-1)!}{2^f}.$$

4. CONCLUSIONS

This paper explores the locating chromatic number of the disjoint union of cycles. This study establishes the disconnected locating chromatic numbers $X'_L(mC_n)$ for $n \in [3, 6]$, $X'_L(mC_{n_i})$ where $n_i \geq 6$ and n_i is even, and maximum m for $X'_L(mC_n) = n$.

5. ACKNOWLEDGEMENT

Thank you for the Hibah Riset Publikasi Bereputasi Universitas Andalas 2022, because has funded this research, enabling it to be successfully completed.

REFERENCES

Abel, L., D. Welyyanti, L. Yulianti, and D. Permana (2025). The Locating Chromatic Number of the Cyclic Chain Graph. *Science and Technology Indonesia*, **10**(3); 958–962

Arawan, A. and A. Istiani (2024). On the Locating-Chromatic Number of the Sunflower Graph. *Sainmatika: Jurnal Ilmiah Matematika dan Ilmu Pengetahuan Alam*, **21**(1); 89–96

Arfin (2025). The Locating-Chromatic Number of Some Jellyfish Graphs. *Journal of the Indonesian Mathematical Society*, **31**(1); 1–10

Asmiati, H. Assiyatun, and E. Baskoro (2011). Locating-Chromatic Number of Amalgamation of Stars. *ITB Journal of Science*, **43A**(1); 1–8

Asmiati and E. Baskoro (2012). Characterizing All Graphs Containing Cycles with Locating-Chromatic Number 3. In *AIP Conference Proceedings*, volume 1451. pages 351–357

Asmiati, E. Baskoro, H. Assiyatun, D. Suprijanto, R. Simanjuntak, and S. Uttunggadewa (2012). The Locating-Chromatic Number of Firecracker Graphs. *Far East Journal of Mathematical Sciences (FJMS)*, **63**(1); 11–23

Asmiati, W. Okzarima, Notiragayu, and L. Zakaria (2024). Upper Bounds of the Locating Chromatic Numbers of Shadow Cycle. *International Journal of Mathematics and Computer Science*, **19**; 239–248

Asmiati, K. Prawinastia, M. Damayanti, and L. Yulianti (2025). The Locating Chromatic Number of (k, n)-Split Cycle Graph and Its Barbell Operation. *Electronic Journal of Graph Theory and Applications*, **13**(2); 271–280

Asmiati, I. K. Sadha Gunce Yana, and L. Yulianti (2018). On the Locating Chromatic Number of Certain Barbell Graphs. *International Journal of Mathematics and Mathematical Sciences*, **2018**(1); 5327504

Asmiati, A., D. Maharani, and Y. Lyra (2021). On the Locating Chromatic Number of Barbell Shadow Path Graphs. *Indonesian Journal of Combinatorics*, **5**(2); 82–93

Assiyatun, H., D. K. Syofyan, and E. T. Baskoro (2020). Locating-Chromatic Number of the Edge-Amalgamation of Trees. *Indonesian Journal of Combinatorics*, **4**(2); 125–131

Baskoro, E. T. and Arfin (2021). All Unicyclic Graphs of Order n with Locating Chromatic Number $n-3$. *Indonesian Journal of Combinatorics*, **5**(2); 73–81

Chartrand, G., D. Erwin, M. Henning, P. Slater, and P. Zhang (2003). Graphs of Order n with Locating-Chromatic Number $n-1$. *Discrete Mathematics*, **269**(1–3); 65–79

Chartrand, G., D. Erwin, M. A. Henning, P. J. Slater, and P. Zhang (2002). The Locating-Chromatic Number of a Graph. *Bull. Inst. Combin. Appl*, **36**(89); 101

Damayanti, M., Asmiati, Fitriani, M. Ansori, and A. Faradilla (2021). The Locating Chromatic Number of Some Modified Path with Cycle Having Locating Number Four. In *Journal of Physics: Conference Series*, volume 1751. IOP Publishing, page 012008

Ghanem, M., H. Al-Ezeh, and A. Dabbour (2019). Locating Chromatic Number of Powers of Paths and Cycles. *Symmetry*, **11**(3); 389

Hamzah, A., Asmiati, and D. W. Amansyah (2024). Locating Chromatic Number for Corona Operation of Path P_n and Cycle C_m, $(m = 3, 4)$. *Indonesian Journal of Combinatorics*, **8**(2); 127–135

Hamzah, N., Asmiati, and A. Nuryaman (2025). The Locating Chromatic Number for Corona Operation of Path and Cycle with Python Programming. *Journal of the Indonesian Mathematical Society*, **31**(4); 1–9

Irawan, A., A. Asmiati, L. Zakaria, and K. Muludi (2021). The Locating-Chromatic Number of Origami Graphs. *Algorithms*, **14**(6); 167

Sakri, R. and M. Abbas (2024). The Locating Chromatic Number of Generalized Petersen Graphs with Small Order. *Examples and Counterexamples*, **5**; 100141

Sakri, R. and B. Slimi (2025). The Bound on the Locating-Chromatic Number for a Generalized Petersen Graphs P (N, 2). *Examples and Counterexamples*, **7**; 100183

Sudarsana, I. W., F. Susanto, and S. Musdalifah (2022). The Locating Chromatic Number for m-Shadow of a Connected Graph. *Electronic Journal of Graph Theory and Applications*, **10**(2); 589–601

Welyyanti, D., L. Abel, and L. Yulianti (2025). The Locating Chromatic Number of Chain$(A, 4, n)$ Graph. *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, **19**(1); 353–360

Welyyanti, D., M. R. Fajri, I. M. Arnawa, I. P. Sandy, W. Arifitriana, L. A. Abel, F. Abbdurrahman, and N. Ghanny (2026a). Locating Chromatic Number of Small Circulant Graph. In *AIP Conference Proceedings*, volume 3389. AIP Publishing LLC, page 020012

Welyyanti, D., R. S. Zahra, and L. Yulianti (2026b). On the Locating-Chromatic Number of the Sunflower Graph. *Journal of the Indonesian Mathematical Society*, **32**(1); 1–12