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Research Paper

Prime and Odd Prime Labelings on Cycle-Related Graphs

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Abstract

Graph labeling is the process of determining integer values for vertices, edges, or both, based on certain criteria. Let G be a simple graph with the finite vertex set $V(G)$. Prime labeling of G is a bijection $\alpha : V(G) \to \{1, 2, \ldots, |V(G)|\}$ for which each pair of adjacent vertices exhibits relatively prime labels. This concept has been extended to odd prime labeling, defined as a bijection $\alpha : V(G) \to \{1, 3, \ldots, 2|V(G)| - 1\}$ satisfying the condition that the labels assigned to adjacent vertices are relatively prime labels. A graph that displays a (odd) prime labeling is designated as a (odd) prime graph. A recent conjecture state that every prime graph is an odd prime graph. In the present study, we conduct an investigation concerning prime and odd prime labeling, focusing on a range of cycle-related graphs classes. Our methods include the axiomatic descriptive approach and pattern detection techniques. We show that volcano graphs, $C_3 \odot_{x_1y_0} F_n$, $C_3 \odot \bar{K}_n$, tadpole graphs, palm trees, and $C_l \odot_{x_1y_0} mP_{n+1}$ are all both prime and odd prime graphs.

Keywords
Cycle-Related Graph, Odd Prime Graph, Relatively Prime

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1. INTRODUCTION

Graph labeling is a mapping from elements of a graph (the domain can be edges, vertices, or both) to integers, subject to specific rules (Wijayanti et al., 2023). Based on its domain, graph labeling is classified into three types: edge labeling, vertex labeling, and total labeling (Gallian, 2024). Graph labeling plays a pivotal role in addressing challenges across numerous fields, such as radio frequency allocation (Griggs and Yeh, 1992), network communications (Prasanna et al., 2014), computer science (Vinutha and Arathi, 2017), cryptography (Prihandoko et al., 2019), crystallography (Kumar and Vats, 2020), and chemistry (Saraswati et al., 2025).

Prime labeling is a type of vertex labeling introduced by Entringer in the early 1980s. It was subsequently popularized by Tout et al. (1982). A graph G of order n admits a prime labeling if there exists a bijection from its vertices to $\{1,2,\ldots,n\}$ such that adjacent vertices receive relatively prime labels. A considerable body of research has demonstrated the existence of prime labelings for numerous classes of trees, including paths, stars, caterpillar with a maximum degree of 5 (Tout et al., 1982); spider graphs, olive tree graphs (Rao, 2002); all tree graphs with a maximum order of 50 (Pikhurko, 2007); palm tree graphs, banana tree graphs, binomial graphs, a number of spider graph families (Robertson and Small, 2009); bistar graphs (Ashokkumar and Maragathavalli, 2015); brush graphs (Samuel and Kalaivani, 2018); and H graphs (Ganesan et al., 2019).

In addition to trees, prime labeling has also been widely studied for graphs containing cycles. Examples of prime cyclic graphs include cycle graphs (Tout et al., 1982); Amal (G,v_0,m) where G is a cycle, a path, or an even wheel (Lee et al., 1988); unions of even cycles (C_{2k}) (Deretsky et al., 1991); fan, helm, flower graphs (Seoud et al., 1998); book graphs (Seoud and Youssef, 1999); unions of cycle and star graphs (Youssef and Elsakhawi, 2007); generalized Petersen graphs (Prajapati and Gajjar, 2015); ladder graphs (Ghorbani and Kamali, 2016); and torch graphs (Wilson and Jini, 2021). Further results on prime labeling can be found in Gallian (2024).

Prime labeling has inspired other types of labeling developed by prior researchers, such as coprime labeling (Berliner et al., 2016), odd prime labeling (Prajapati and Shah, 2018), relative prime edge labeling (Janani and Ramachandran, 2022), and others. An overview of the development of prime labeling can be found in (Gallian, 2024). Odd prime labeling on a graph G is development of prime labeling, introduced by Prajapati and Shah (2018). It is defined as a bijective function that maps the vertices $V(G)$ onto the set of odd number from 1 to $(2|V(G)| - 1)$, such that the labels of adjacent vertices are relatively prime. Graphs that admit such a labeling are called odd prime graphs. Prajapati and Shah (2018) advanced the conjecture that every prime graph is necessarily an odd prime graph. This conjecture has been the subject of extensive research, with many scholars reviewing and analyzing its implications. Prajapati and Shah (2018) demonstrated that several classes of prime graphs are also odd prime graphs, including path, fan, ladder, wheel, complete bipartite, cycle, gear, flower, closed helm, helm, and generalized Petersen $P_{n,2}$ graphs. Several other prime graphs have since been shown to be odd prime graphs, including union of paths (Youssef and Almoreed, 2020), graphs $\bigcup_{i=1}^{n} C_{k_i}$ where $k \equiv 0 \pmod 2$, snake, prism, spider, binary tree, caterpillar with a maximum degree of 5, firecracker, P_n^2, C_n^2, the prism $GP_{(n,1)}$ (Carter and Fox, 2022), and circular ladder (Meena and Gajalakshmi, 2022).

Recent studies on the odd prime labeling over the past three to five years have primarily focused on ladder graph, prism graph, spider graph, and other families of structured graph with relatively regular structures (Youssef and Almoreed, 2020; Carter and Fox, 2022; Meena and Gajalakshmi, 2022). However, there has been limited research on union graphs where cycles are attached to stars or fans and the operation of cycles with null graphs through the corona process. These graphs exhibit richer structural behavior than leaf-free cycle families, rendering them significant testing grounds for constructing prime and odd-prime labelings.

In this paper, we address this gap by investigating prime and odd prime labeling on several cycle-related union graphs, namely the volcano, $C_3 \odot_{x_1\ell_0} F_n$, $C_3 \odot \bar{K}_n$, tadpole, palm tree, and $C_l \odot_{x_1y_0} mP_{n+1}$. These graph families are selected not merely because they contain cycles, but they also represent fundamentally different mechanisms of combining cycles with trees. Specifically, the volcano attaches a star to a cycle, the tadpole attaches a path to a cycle, and the palm tree combines both graphs into a branched cycle structure. This diversity enables a systematic examination of how different types of cycle attachment influence the existence of prime and odd prime labeling.

To the best of our knowledge, prime and odd prime labeling for these classes of graphs have not been previously established. The primary contributions of this work are as follows: (i) we provide explicit constructions for prime and odd prime labeling on these previously unexplored graph families, and (ii) we develop generalized labeling patterns that extend existing techniques beyond standard cyclic graphs such as cycles, wheels, and ladders. While our results provide further evidence supporting the conjecture that every prime graph is an odd prime graph, this support remains limited to the specific classes discussed in this paper. Therefore, our findings should be regarded as partial progress rather than a general confirmation of the conjecture, thereby encouraging further investigation into broader and more complex graph families.

2. EXPERIMENTAL SECTION

We employ two research methods, incorporating the axiomatic descriptive method and the pattern detection method. The axiomatic descriptive method consists of reviewing several lemmas related to prime or odd prime labeling. The pattern detection method involves the analysis of patterns in graphs of a specific order, which are subsequently generalized into a theorem. The following are several lemmas about some conditions of two positive integers, both of each pair being relatively primes to the other.

Lemma 2.1. *Suppose that k and ℓ are positive integers.*
- (a) *If k and ℓ are consecutive, then k and ℓ are relatively prime Sukirman (2016).*
- (b) *If k is odd, then there exists a positive integer m which has no odd factors other than one such that $\gcd(k, k+m) = 1$ Komarullah et al. (2022).*

Next, we present several definitions pertinent to this study.

Definition 2.2. *Let G_1 and G_2 be graphs, with $x \in V(G_1)$ and $y \in V(G_2)$. The graph $G_1 \odot_{x,y} G_2$ is a graph acquired from two disjoint graphs G_1 and G_2 by identifying vertices $x \in V(G_1)$ and $y \in V(G_2)$. Thus, the graph $G_1 \odot_{x,y} G_2$ has $(|V(G_1)| + |V(G_2)| - 1)$ vertices and $(|E(G_1)| + |E(G_2)|)$ edges (Borowiecka-Olszewska and Hałuszczak, 2013). The identification operation on graphs G_1 and G_2 is commutative, such that $G_1 \odot_{x,y} G_2 = G_2 \odot_{y,x} G_1$.*

An example of the vertex identification operation between G_1 and G_2 is shown in Figure 1. The vertex identification operation between G_1 and G_2 on vertices $x_1 \in G_1$ and $y_4 \in G_2$, denoted by $(G_1 \odot_{x_1,y_4} G_2$ or $G_1 \odot_{y_4,x_1} G_2)$, can be seen in Figure 1(c).

Definition 2.3. *Let G_1 and G_2 be two graphs. The corona of G_1 and G_2, represented by $G_1 \odot G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and by joining each vertex of the i^{th} copy of G_2 to the i^{th} vertex of G_1, where $1 \le i \le |V(G_1)|$ (Frucht and Harary, 1970). The corona product of G_1 and G_2 is not commutative, such that $G_1 \odot G_2 \ne G_2 \odot G_1$.*

For example, Figure 2 illustrates the corona product of a cycle (C_3) and a path (P_4). Figure 2(a) depicts $P_4 \odot C_3$ and Figure 2(b) shows $C_3 \odot P_4$.

Definition 2.4. *A graph star with $n+1$ vertices, denoted by S_n, is a tree graph that has one vertex of degree n and n other vertices of degree 1 (Diestel, 2025). A fan graph with $n+1$ vertices, denoted by F_n, is a graph constructed from a path P_n and $\overline{K}_1$ by connecting each vertex of P_n to the vertex of $\overline{K}_1$ (Bača et al., 2021).*

3. RESULT AND DISCUSSIONS

In the subsequent section, we delve into the discussion of prime and odd prime labeling on several classes of cycle-related graphs namely volcano, $C_3 \odot_{x_1y_0} F_n$, $C_3 \odot \bar{K}_n$, tadpole, palm tree, and $C_l \odot_{x_1y_0} mP_{n+1}$.

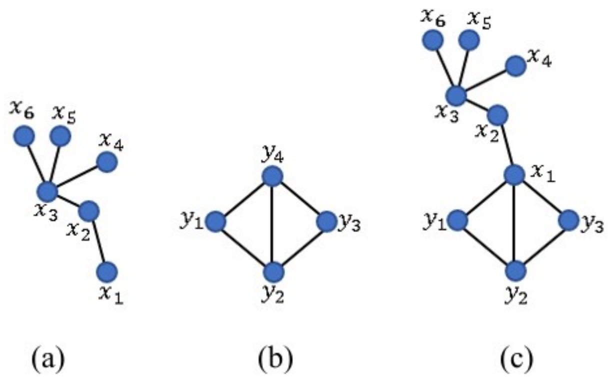

Figure 1. G_1 (a), G_2 (b), and $G_1 \odot_{x_1, y_4} G_2$ (c)

3.1 Volcano Graph V_n and Related Graphs

A volcano graph V_n refers to a graph comprising $n + 3$ vertices and edges that is isomorphic to $C_3 \odot_{x_1 y_0} S_n$ where $x_1 \in V(C_3)$ and $y_0 \in S_n$. Specifically, the center of the star graph, which corresponds to the vertex with the highest degree, is identified to one of the vertices in the cycle graph C_3 (Dafik et al., 2023). To discuss prime and odd prime labeling on volcano graphs, we first present the set of vertices and edges of the volcano graph, focusing on the key elements necessary for the subsequent content.

$$V(V_n) = \{x_k | k = 1, 2, 3\} \cup \{y_\ell | \ell = 1, 2, \dots, n\}.$$
$$E(V_n) = \{x_1 x_3\} \cup \{x_k x_{k+1} | k = 1, 2\} \cup \{x_1 y_\ell | \ell = 1, 2, \dots, n\}.$$

Theorem 3.1. *For every $n \geq 1$, the volcano graph (V_n) is a prime graph.*

To show that the volcano graph (V_n) is a prime graph, we construct a prime labeling on the graph. The function $\alpha : V(V_n) \to \{1, 2, \dots, n + 3\}$ is defined as follows:

$$\alpha(x_k) = k, \quad \text{for } k = 1, 2, 3,$$
$$\alpha(y_\ell) = \ell + 3, \quad \text{for } \ell = 1, 2, \dots, n.$$

According to the principles of prime labeling, any two adjacent vertices must be assigned relatively prime labels. Based on Lemma 2.1(a), the labels of vertices x_k and x_{k+1} are relatively prime due to their consecutive disposition. The labels of the vertex x_1 with x_3 and x_1 with y_ℓ are also relatively prime, since $\alpha(x_1) = 1$ is relatively prime to all positive integers. Therefore, the function α satisfies the rule of prime labeling, and it follows that the volcano graph (V_n) is a prime graph.

Theorem 3.2. *For every $n \geq 1$, the volcano graph (V_n) is an odd prime graph.*

To show that the volcano graph (V_n) is an odd prime graph, the function $\alpha : V(V_n) \to \{1, 3, \dots, 2n + 5\}$ is defined as follows:

$$\alpha(x_k) = 2k - 1, \quad \text{for } k = 1, 2, 3,$$
$$\alpha(y_\ell) = 2\ell + 5, \quad \text{for } \ell = 1, 2, \dots, n.$$

It can be readily confirmed that each pair of adjacent vertices receives labels that are relatively prime. Therefore, the volcano graph (V_n) qualifies as an odd prime graph.

Example 3.3. Prime and odd prime labeling on the volcano graph can be seen in both Figure 3(a) and Figure 3(b), accordingly.

Moreover, we can construct the related graph from the volcano graph by connecting vertices y_ℓ and $y_{\ell+1}$ with $\ell = 1, 2, \dots, n - 1$ in order for it to become the graph $V_n + y_\ell y_{\ell+1}$ that is isomorphic to the graph $C_3 \odot_{x_1 y_0} F_n$.

Corollary 3.4. *For every $n \geq 1$, the graph $C_3 \odot_{x_1 y_0} F_n$ is prime and also odd prime.*

Proof. In accordance with Theorem 3.1, it is apparent that $\alpha(y_\ell)$ and $\alpha(y_{\ell+1})$ with $\ell = 1, 2, \dots, n - 1$ are consecutive, as supported by Lemma 2.1(a), so $V_n + y_\ell y_{\ell+1}$ has labels for each vertex that are relatively prime. Based on Theorem 3.2, it is known that $|\alpha(y_\ell) - \alpha(y_{\ell+1})| = 2$, since 2 has no other odd factor besides 1. Thus, by Lemma 2.1(b), the labels of vertices y_ℓ and $y_{\ell+1}$ are relatively prime.

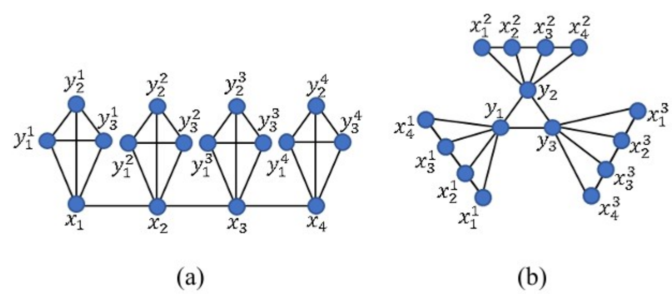

Figure 2. $P_4 \odot C_3$ (a) and $C_3 \odot P_4$ (b)

Next, suppose a new graph is generated through the identification of two star graphs S_n and by attaching their center vertices to x_2 and x_3, thereby forming a new graph $C_3 \odot \overline{K}_n$, in which $\overline{K}_n$ is a null graph with n vertices. The collection of vertices and edges in the graph $C_3 \odot \overline{K}_n$ is demonstrated as follows:

$$V(C_3 \odot \overline{K}_n) = \{x_k \mid k = 1, 2, 3\} \cup \{y_\ell^k \mid \ell = 1, 2, \dots, n;\ k = 1, 2, 3\},$$

$$E(C_3 \odot \overline{K}_n) = \{x_1 x_3\} \cup \{x_k x_{k+1} \mid k = 1, 2\} \cup \{x_k y_\ell^k \mid \ell = 1, 2, \dots, n;\ k = 1, 2, 3\}.$$

Example 3.5. The notation for vertices and edges in the graph $C_3 \odot \overline{K}_n$ can be seen in Figure 4.

Theorem 3.6. *For every $n \geq 2$, the graph $C_3 \odot \overline{K}_n$ is a prime graph.*

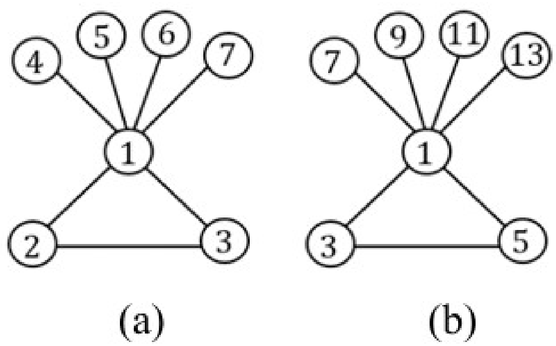

Figure 3. Prime (a) and Odd Prime (b) Labeling on Volcano Graph (V_4)

Proof. In a similar manner, a prime labeling is constructed on the graph $C_3 \odot \overline{K_n}$. Define a function $\alpha : V(C_3 \odot \overline{K_n}) \to \{1, 2, \ldots, 3n+3\}$ as follows:

$\alpha(x_k) = k$, for $k = 1, 2,$ and 3.

$$\alpha(y_\ell^1) = \begin{cases} 3\ell+3, & \text{for } \ell \text{ is odd,} \\ 3\ell+2, & \text{for } \ell \text{ is even.} \end{cases}$$

$$\alpha(y_\ell^2) = \begin{cases} 3\ell+2, & \text{for } \ell \text{ is odd,} \\ 3\ell+3, & \text{for } \ell \text{ is even.} \end{cases}$$

$\alpha(y_\ell^3) = 3\ell+1$, for $\ell = 1, 2, \ldots, n$.

We now verify that every pair of adjacent vertices has labels that are relatively prime.

1. $\gcd(\alpha(y_\ell^2), \alpha(x_2)) = \gcd(3\ell+2, 2) = 1$, since $3\ell+2$ is odd when ℓ is odd.
2. $\gcd(\alpha(y_\ell^2), \alpha(x_2)) = \gcd(3\ell+3, 2) = 1$, since $3\ell+3$ is odd when ℓ is even.
3. $\gcd(\alpha(y_\ell^3), \alpha(x_3)) = \gcd(3\ell+1, 3) = 1$, since $3\ell+1 \neq 0 \bmod 3$ for any ℓ.

Therefore, all adjacent vertex pairs have relatively prime labels, and $C_3 \odot \overline{K_n}$ is a prime graph.

Theorem 3.7. *For every $n \geq 2$, the graph $C_3 \odot \overline{K_n}$ is an odd prime graph.*

Proof. Assuming α is a labeling of the vertices of $C_3 \odot \overline{K_n}$ with labels from $\{1, 2, \ldots, 6n+5\}$, the latter is defined as follows:

$\alpha(x_k) = 2k-1$, for $k = 1, 2, 3$.

$$\alpha(y_\ell^1) = \begin{cases} 6\ell+3, & \text{for } \ell \not\equiv 0 \pmod 5, \\ 6\ell+5, & \text{for } \ell \equiv 0 \pmod 5. \end{cases}$$

$\alpha(y_\ell^2) = 6\ell+1$, for $\ell = 1, 2, \ldots, n$.

$$\alpha(y_\ell^3) = \begin{cases} 6\ell+5, & \text{for } \ell \not\equiv 0 \pmod 5, \\ 6\ell+3, & \text{for } \ell \equiv 0 \pmod 5. \end{cases}$$

The proof that each adjacent label has a relatively prime label is explained as follows:

a. $\gcd(\alpha(x_2), \alpha(y_\ell^2)) = \gcd(3, 6\ell+1) = 1$, since $6\ell+1 \equiv 1 \pmod 3$.

b. $\gcd(\alpha(x_3), \alpha(y_\ell^3)) = \gcd(5, 6\ell+5) = 1$, since $\ell \not\equiv 0 \pmod 5$, label $6\ell+5 \not\equiv 0 \pmod 5$.

c. $\gcd(\alpha(x_3), \alpha(y_\ell^3)) = \gcd(5, 6\ell+3) = 1$, since $\ell \equiv 0 \pmod 5$, label $6\ell+3 \not\equiv 0 \pmod 5$.

We have proven that for any two adjacent vertices, the labels are relatively prime. So, the graph $C_3 \odot \overline{K_n}$ is an odd prime graph.

3.2 Tadpole Graphs ($T_{m,n}$)

In the following subsection, we engage in a discussion of prime and odd prime on the tadpole graph. The tadpole graph $T_{m,n}$ in which $m \geq 3$ and $n \geq 2$ is a graph that possesses an order and size of $m+n$. It is constructed from the identification of the vertices on the cycle graph C_m and the path graph P_{n+1} by identifying the vertex $x_1 \in V(C_m)$ with the vertex $y_0 \in V(P_{n+1})$ or we can write it as $C_m \odot_{x_1 y_0} P_{n+1}$ (DeMaio and Jacobson, 2014). To discuss the prime and odd prime labeling on the tadpole graph, we first establish the notation for vertices and edges in the tadpole graph.

$$V(T_{m,n}) = \{x_k \mid k = 1, 2, \ldots, m\} \cup \{y_\ell \mid \ell = 1, 2, \ldots, n\}.$$

$$E(T_{m,n}) = \{x_1 y_1\} \cup \{x_k x_{k+1}, x_1 x_m \mid k = 1, 2, \ldots, m-1\} \cup \{y_\ell y_{\ell+1} \mid \ell = 1, 2, \ldots, n-1\}.$$

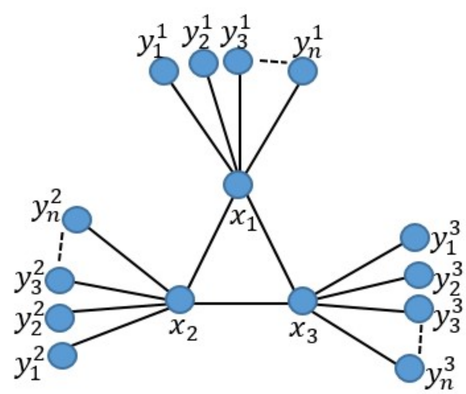

Figure 4. The Graph $C_3 \odot \overline{K_n}$

Theorem 3.8. *For every $m \geq 3$ and $n \geq 2$, tadpole graph $T_{m,n}$ is a prime graph.*

Proof. To show that the tadpole graph $T_{m,n}$ is a prime graph, we construct a prime labeling on the graph. The function $\alpha : V(T_{m,n}) \to \{1, 2, \ldots, m+n\}$ is formulated as follows:

$\alpha(x_k) = k$, for $k = 1, 2, \ldots, m$.

$\alpha(y_\ell) = m+\ell$, for $\ell = 1, 2, \ldots, n$.

We have established that integer 1 is relatively prime to all integers, hence it follows that $\gcd(\alpha(x_1), \alpha(x_2)) = \gcd(\alpha(x_1), \alpha(x_m)) = \gcd(\alpha(x_1), \alpha(y_1)) = 1$. Furthermore, the labels of vertices x_k and x_{k+1}, as well as y_ℓ and $y_{\ell+1}$ are consecutive, thus by Lemma 2.1(a) $\gcd(\alpha(x_k), \alpha(x_{k+1})) = \gcd(\alpha(y_\ell), \alpha(y_{\ell+1})) = 1$. This proves that the labels of any two adjacent vertices are relatively prime. Consequently, the function α follows the prime labeling rule, which signifies that the tadpole graph $T_{m,n}$ is a prime graph.

Theorem 3.9. *For every $m \geq 3$ and $n \geq 2$, tadpole graph $(T_{m,n})$ is an odd prime graph.*

Proof. In the same way, to show that the graph tadpole graph $(T_{m,n})$ is an odd prime graph, we construct an odd prime labeling on the graph. The function $\alpha : V(T_{m,n}) \to \{1, 2, \ldots, 2(m+n) - 1\}$ is defined in the following.

$\alpha(x_k) = 2k - 1, \quad$ for $k = 1, 2, \ldots, m$.

$\alpha(y_\ell) = 2(m + \ell) - 1, \quad$ for $\ell = 1, 2, \ldots, n$.

According to the prime labeling rule, any adjacent labels are relatively prime. We demonstrate the label as follows.

a. We will show that the labels of vertices x_k and x_{k+1} are relatively prime. It was found that $|\alpha(x_k) - \alpha(x_{k+1})| = 2$. Since 2 does not have any odd factors other than 1, hence, by Lemma 2.1(b) we have $\gcd(x_k, x_{k+1}) = 1$.
b. Furthermore, we also show that the labels of vertices y_ℓ and $y_{\ell+1}$ are relatively prime. It is known that $|\alpha(y_\ell) - \alpha(y_{\ell+1})| = \left|(2(m+\ell) - 1) - (2(m+\ell+1) - 1)\right| = 2$. Since 2 does not have any odd factors other than 1, hence, by Lemma 2.1(b) we have $\gcd(y_\ell, y_{\ell+1}) = 1$.

Thus, it is proved that the function α satisfies the odd prime labeling rule, which means that the tadpole graph $(T_{m,n})$ is an odd prime graph.

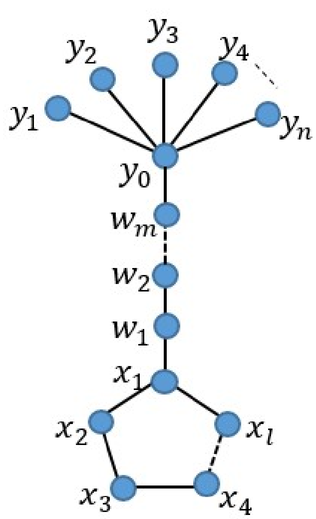

Figure 5. The Palm Tree Graph $(C_l P_m S_n)$

3.3 Palm Tree Graphs $(C_l P_m S_n)$

The palm tree graph with order $l + m + n + 1$ is denoted by ClP_mS_n. It is constructed by performing a vertex identification between the tadpole graph $T_{l,m+1}$ and the star graph S_n, by attaching the end vertices of the path in tadpole graph to the center of star graph S_n (Mujib, 2019). The palm tree graph ClP_mS_n has the following sets of vertices and edges.

$$V(ClP_mS_n) = \{x_k \mid k = 1, 2, \ldots, l\} \cup \{w_\ell \mid \ell = 1, 2, \ldots, m\} \cup \{y_m \mid m = 0, 1, \ldots, n\}.$$

$$E(ClP_mS_n) = \{x_k x_{k+1}, x_1 x_l \mid k = 1, 2, \ldots, l-1\} \cup \{x_1 w_1\} \cup \{w_\ell w_{\ell+1} \mid \ell = 1, 2, \ldots, m-1\} \cup \{w_m y_0\} \cup \{y_0 y_m \mid m = 1, 2, \ldots, n\}.$$

Example 3.10. The notation of vertices and edges on the palm tree graph can be observed in Figure 5.

Theorem 3.11. *For every $l \geq 3$, $m \geq 3$ and $n \geq 2$, palm tree graph ClP_mS_n is a prime graph.*

Proof. To demonstrate that the palm tree graph ClP_mS_n is a prime graph, we introduce a prime labeling for its vertices. Let $\alpha : V(ClP_mS_n) \to \{1, 2, \ldots, l+m+n+1\}$, it can be calculated as follows.

a. Case l even

$\alpha(x_k) = k + 1, \quad$ for $k = 1, 2, \ldots, l$.

$\alpha(w_\ell) = l + \ell + 2, \quad$ for $\ell = 1, 2, \ldots, m$.

$$\alpha(y_m) = \begin{cases} 1, & \text{for } m = 0, \\ l + 2, & \text{for } m = 1, \\ m + m + l + 1, & \text{for } 2 \leq m \leq n. \end{cases}$$

We will demonstrate that the labels of any two adjacent vertices are relatively prime.

i. Given that $\alpha(x_1) = 2$, and since 2 is relatively prime to all odd integers, it follows that $\gcd(\alpha(x_1), \alpha(x_2)) = \gcd(\alpha(x_1), \alpha(x_l)) = \gcd(\alpha(x_1), \alpha(w_1)) = 1$.
ii. According to Lemma 2.1(a), $\gcd(\alpha(x_k), \alpha(x_{k+1})) = \gcd(\alpha(w_\ell), \alpha(w_{\ell+1})) = 1$ because these labels are consecutive.
iii. The label of vertex $\alpha(y_0) = 1$, and the integer 1 is relatively prime to all integers. Thus, $\gcd(\alpha(y_0), \alpha(y_m)) = \gcd(\alpha(y_0), \alpha(w_m)) = 1$.

b. Case l odd

$$\alpha(x_k) = \begin{cases} k + 1, & \text{for } k = 1, 2, \ldots, l-1, \\ l + 2, & \text{for } k = l. \end{cases}$$

$\alpha(w_\ell) = l + \ell + 3, \quad$ for $\ell = 1, 2, \ldots, m$.

$$\alpha(y_m) = \begin{cases} 1, & \text{for } m = 0, \\ 2m + l - 1, & \text{for } m = 1 \text{ and } 2, \\ m + m + l + 1, & \text{for } m = 3, 4, \ldots, n. \end{cases}$$

In this case, we will also demonstrate that the labels of all adjacent vertices are relatively prime.

a. We have determined that integer 2 is relatively prime to all integers, thereby resulting in $\gcd(\alpha(x_1), \alpha(x_2)) = \gcd(\alpha(x_1), \alpha(x_l)) = \gcd(\alpha(x_1), \alpha(w_1)) = 1$.

b. According to Lemma 2.1(a), for every $k = 2, 3, \ldots, l-2$, the labels of vertices x_k and x_{k+1} are relatively prime.
c. The labels of vertices w_ℓ and $w_{\ell+1}$ are also relatively prime due to their consecutive labeling pattern.
d. Given that $\alpha(y_0) = 1$, it follows that $\gcd(\alpha(y_0), \alpha(w_m)) = \gcd(\alpha(y_0), \alpha(y_m)) = 1$, as the integer 1 is relatively prime to any integer.

Based on the demonstration above, the labels of all adjacent vertices are relatively prime. Therefore, it is evident that the palm tree graph ClP_mS_n is a prime graph.

Theorem 3.12. *For every $l \geq 3$, $m \geq 3$ and $n \geq 2$, palm tree graph ClP_mS_n is an odd prime graph.*

Proof. It is shown that the palm tree graph ClP_mS_n is an odd prime graph, we construct an odd prime labeling on the graph. The function $\alpha : V(ClP_mS_n) \to \{1, 2, \ldots, 2(l+m+n)+1\}$ is calculated as follows.

a. Case $l \equiv 0 \pmod 3$

$\alpha(x_k) = 2k+1, \quad \text{for } k = 1, 2, \ldots, l.$

$\alpha(w_\ell) = 2(l+\ell)+3, \quad \text{for } \ell = 1, 2, \ldots, m.$

$$\alpha(y_m) = \begin{cases} 1, & \text{for } m = 0, \\ 2l+3, & \text{for } m = 1, \\ 2(m+m+l)+1, & \text{for } m = 2, 3, \ldots, n. \end{cases}$$

In order to show that the function α satisfies the odd prime labeling criteria, we must demonstrate that the labels of all adjacent vertices are relatively prime.

i. Given that for every $k = 1, 2, \ldots, l-1$, the value of $|\alpha(x_k) - \alpha(x_{k+1})| = 2$. Based on Lemma 2.1(b), $\gcd(\alpha(x_k), \alpha(x_{k+1})) = 1$ since 2 has no odd factor other than 1.
ii. The value of $\alpha(x_1) \equiv 1 \pmod 3$ and $\alpha(w_1) \equiv 2 \pmod 3$, then it follows that $\gcd(\alpha(x_1), \alpha(x_l)) = \gcd(\alpha(x_1), \alpha(w_1)) = 1$.
iii. For every $\ell = 1, 2, \ldots, m-1$, the value of $|\alpha(w_\ell) - \alpha(w_{\ell+1})| = 2$. Integer 2 only has an odd factor of 1, thus by Lemma 2.1(b) $\gcd(\alpha(w_\ell), \alpha(w_{\ell+1})) = 1$.
iv. Integer 1 is relatively prime to all integers, such that $\gcd(\alpha(y_0), \alpha(y_m)) = \gcd(\alpha(y_0), \alpha(w_\ell)) = 1$.

b. Case $l \equiv 1 \pmod 3$

$$\alpha(x_k) = \begin{cases} k+1, & \text{for } k = 1, 2, \ldots, l-1, \\ 2l+3, & \text{for } k = l. \end{cases}$$

$\alpha(w_\ell) = 2(l+\ell)+3, \quad \text{for } \ell = 1, 2, \ldots, m.$

$$\alpha(y_m) = \begin{cases} 1, & \text{for } m = 0, \\ 2l+1, & \text{for } m = 1, \\ 2(l+m+m)+1, & \text{for } m = 2, 3, \ldots, n. \end{cases}$$

In a similar manner, it can be demonstrated that the function α satisfies the odd prime labeling criteria. The value of $\alpha(x_1) = 3$, $\alpha(x_l) \equiv 2 \pmod 3$, and $\alpha(w_1) \equiv 1 \pmod 3$. These shows that $\gcd(\alpha(x_1), \alpha(x_l)) = \gcd(\alpha(x_1), \alpha(w_1)) = 1$.

c. Case $l \equiv 2 \pmod 3$

$\alpha(x_k) = 2k+1, \quad \text{for } k = 1, 2, \ldots, l.$

$\alpha(w_\ell) = 2(l+\ell)+1, \quad \text{for } \ell = 1, 2, \ldots, m.$

$$\alpha(y_m) = \begin{cases} 1, & \text{for } m = 0, \\ 2(m+m+l)+1, & \text{for } m = 1, 2, \ldots, n. \end{cases}$$

In accordance with the proof of case (b), the value of $\alpha(x_1) = 3$, $\alpha(x_l) \equiv 2 \pmod 3$, and $\alpha(w_1) \equiv 1 \pmod 3$ yields the $\gcd(\alpha(x_1), \alpha(x_l)) = \gcd(\alpha(x_1), \alpha(w_1)) = 1$.

It is clear that the labels for every pair of adjacent vertices are relatively prime. So, this proves that the palm tree graph $Cl{-}B_{m,n}$ is an odd prime graph.

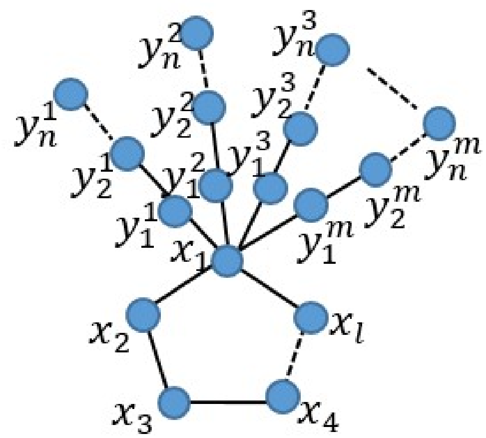

Figure 6. The Notation of Vertices and Edges on the Graph $C_l \odot_{x_1,y_0} mP_{n+1}$

3.4 Graph Formed by the Vertex Identification ($C_l \odot_{x_1 y_0} mP_{n+1}$)

The graph formed by the vertex identification $C_l \odot_{x_1,y_0} mP_{n+1}$ is obtained by attaching the end vertices, which in this case is the vertex v_0 from the m duplicate of the path graph (P_n) to the vertex u_1 in the cycle graph (C_l). The set of vertices and edges in the graph formed by this vertex identification $C_l \odot_{x_1,y_0} mP_{n+1}$ is as follows.

$$V(C_l \odot_{x_1,y_0} mP_{n+1}) = \{x_k \mid k = 1, 2, \ldots, l\} \cup \{y_\ell^i \mid i = 1, 2, \ldots, m;\ \ell = 1, 2, \ldots, n\}$$

$$E(C_l \odot_{x_1,y_0} mP_{n+1}) = \{x_1 x_l\} \cup \{x_k x_{k+1} \mid k = 1, 2, \ldots, l-1\} \cup \{x_1 y_1^i \mid i = 1, 2, \ldots, m\} \cup \{y_\ell^i y_{\ell+1}^i \mid i = 1, 2, \ldots, m;\ \ell = 1, 2, \ldots, n-1\}$$

Example 3.13. The notation of vertices and edges on the graph $C_l \odot_{x_1,y_0} mP_{n+1}$ is illustrated in Figure 6.

Theorem 3.14. *For every $m \geq 1$, $n \geq 2$, and $l \geq 3$, graph $C_l \odot_{x_1,y_0} mP_{n+1}$ is a prime graph.*

Proof. To show that the graph $C_l \odot_{x_1,y_0} mP_{n+1}$ is a prime graph, we construct a prime labeling on the graph. The function $\alpha : V(C_l \odot_{x_1,y_0} mP_{n+1}) \to \{1, 2, \ldots, mn + l\}$ is defined as follows:

$\alpha(x_k) = k,$ for $k = 1, 2, \ldots, l.$

$\alpha(y^i_\ell) = l + (i - 1)n + \ell,$ for $i = 1, 2, \ldots, m;$ $\ell = 1, 2, \ldots, n.$

The label of vertex $x_1 = 1$, resulting in $\gcd(\alpha(x_1), \alpha(x_l)) = \gcd(\alpha(x_1), \alpha(y^i_1)) = 1$ because 1 is relatively prime to all integers. Furthermore, the labels of vertices x_k and x_{k+1} are relatively prime based on Lemma 2.1(a) as they are consecutive. Similarly, the labels of vertices y^i_ℓ and $y^i_{\ell+1}$ are relatively prime because they are consecutive integers.

Since the function α distributes relatively prime labels to every pair of adjacent vertices, we confirm that the graph $C_l \odot_{x_1,y_0} mP_{n+1}$ is a prime graph.

Theorem 3.15. *For every $m \geq 1$, $n \geq 2$, and $l \geq 3$, graph $C_l \odot_{x_1,y_0} mP_{n+1}$ is an odd prime graph.*

Proof. In the same way, to show that the graph $C_l \odot_{x_1,y_0} mP_{n+1}$ is an odd prime graph, we construct an odd prime labeling on the graph.

The function $\alpha : V(C_l \odot_{x_1,y_0} mP_{n+1}) \to \{1, 3, \ldots, 2(mn+l) - 1\}$ is defined as follows:

$\alpha(x_k) = 2k - 1,$ for $k = 1, 2, \ldots, l.$

$\alpha(y^i_\ell) = 2(l + \ell) + 2n(i - 1) - 1,$ for $i = 1, 2, \ldots, m;$ $\ell = 1, 2, \ldots, n.$

Given that $|\alpha(x_k) - \alpha(x_{k+1})| = 2$, it follows from Lemma 2.1(b) that $\gcd(\alpha(x_k), \alpha(x_{k+1})) = 1$ since 2 has no odd factors other than 1. In a similar manner, we obtain $|\alpha(y^i_\ell) - \alpha(y^i_{\ell+1})| = 2$, which implies that $\gcd(\alpha(y^i_\ell), \alpha(y^i_{\ell+1})) = 1$.

We observe that function α assigns labels that are relatively prime to every pair of adjacent vertices. Therefore, it has been proven that the graph $C_l \odot_{x_1,y_0} mP_{n+1}$ is an odd prime graph.

4. CONCLUSIONS

On the basis of the research findings, we can prove that volcano graphs, $C_3 \odot_{x_1,y_0} F_n$, $C_3 \odot \overline{K_n}$, tadpole graphs, palm tree graphs, and $C_l \odot_{x_1,y_0} mP_{n+1}$ are prime graphs and odd prime graphs. These results support the conjecture stating prime graphs are indeed odd prime graphs for specific class graphs. Nevertheless, the validity of this conjecture remains elusive on a general scale. Therefore, the authors would like to suggest future researchers for further investigation of other prime graphs and the implementation of odd prime labeling in the graphs. Alternatively, the conjecture can be proven in general.

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