Science and Technology Indonesia
e-ISSN:2580-4391 *p*-ISSN:2580-4405
Vol. 11, No. 2, April 2026

Research Paper

Non-Inclusive and Inclusive Distance Irregularity Strength of Complement and Split Graphs

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Abstract
Let f be a map from vertices of a graph G to number from 1 to k. The labeling f is a distance irregular labeling (DIL) if for every two vertices x and y, we have $wt_f(u) \neq wt_f(v)$ where a weight $wt_f(u)$ is defined as the sum of labels of the neighbors of u. Moreover, the labeling f is a inclusive distance irregular labeling (IDIL) if for every two vertices x and y, $wt_f(u) \neq wt_f(v)$ with a weight $wt_f(u)$ is defined as the sum of the label of u and the labels of the neighbors of u. The least number k where there exists a DIL is called a distance irregularity strength (DIS), denoted by $\text{dis}(G)$. Similarly, the least number k where there exists a IDIL is called an inclusive distance irregularity strength (IDIS), denoted by $\widehat{\text{dis}}(G)$. In this study, we present a connection of DIL and IDIL in a graph with its complement. In particular, we derive a new upper bound for DIS and IDIS of any graph. Further, we determine the $\text{dis}(G)$ and $\widehat{\text{dis}}(G)$ for certain special family of split graph G and provide examples of a graph G satisfying $\text{dis}(G) = \widehat{\text{dis}}(G)$.

Keywords
Distance Irregularity Strength, Inclusive Distance Irregularity Strength, Complement of Graphs, Split Graphs

Received: 11 November 2025, Accepted: 3 March 2026
https://doi.org/10.26554/sti.2026.11.2.652-660

1. INTRODUCTION

In 1988, Chartrand et al. (1988) introduced the concept of irregular networks, laying the foundation for studying graph irregularity. For a simple and finite graph G, let $f : E(G) \to [1, k]$ be an edge k-labeling of G. The labeling f is called *irregular* if for every two distinct vertices u and v it holds that $wt_f(u) \neq wt_f(v)$, where $wt_f(u) = \sum_{ux\in E(G)} f(ux)$ with the sum is taken over all edges incident to u. The *irregularity strength* of G, denoted by $s(G)$, is the least positive integer k where there exists an irregular labeling f for G (Chartrand et al., 1988). Faudree et al. (1989) determined the irregularity strength of regular graphs. The study of irregular graphs also has been conducted to dense graphs (Majerski and Przybyło, 2014; Przybyło, 2025). Moreover, Bača et al. (2021) investigated the irregularity strength and its modular version of fan graphs. It is found that problems similar to determining the irregularity strength of graphs are NP-hard (Bensmail, 2022).

Over time, there are several variations of irregular labelings considered by many. One of them is called edge irregularity labeling introduced by Ahmad et al. (2014). For a positive integer k, a k-labeling $f : V(G) \to [1, k]$ is said to be *edge irregular* if for every two distinct edges uv and xy, $wt_f(uv) \neq wt_f(xy)$, where $wt_f(uv) = f(u)+f(v)$ for every edge $uv \in E(G)$. Denoted by $es(G)$, the minimum number k such that there exists a k-labeling of the graph G is said to be the *edge irregularity strength* of G. Then, Ashraf et al. (2016) take a step further by coined the notion of vertex H-irregular labeling. A graph G is said to admit *H-covering* if for every edge $e \in E(G)$, there exists a subgraph of G which is isomorphic to H and the edge e is contained in H. For a graph G that admits H-covering, a map $f : V(G) \to [1, k]$ is said to be *vertex H-irregular* if for every two distinct subgraphs H_1 and H_2 isomorphic to H we have.

$$wt_f(H_1) = \sum_{v\in V(H_1)} f(v) = \sum_{v\in V(H_2)} f(v) = wt_f(H_2). \qquad (1)$$

The *vertex H-irregularity strength* of a graph G is the least positive number k for which G has a vertex H-irregularity labeling, which is denoted by $vhs(G, H)$. We can consider that the notion of vertex H-irregularity strength generalized the edge irregularity strength, since $vhs(G, K_2) = es(G)$ for every graph G.

Furthermore, Slamin (2017) coined the notion of (non-inclusive) distance irregular labeling of graphs. A k-labeling $f : V(G) \to [1, k]$ is a *distance irregular labeling* (DIL) if for every two distinct vertices u and v, it follows that $w(u) \neq w(v)$ given that $w(u) = \sum_{x \in N_G(u)} f(x)$ (if $N_G(u) = \varnothing$, then $wt_f(u) = 0$). The *distance irregularity strength* (DIS), $\mathrm{dis}(G)$, of G is the least integer k where G has a distance irregular k-labeling (Slamin, 2017). The study of DIS-ness of graphs has been conducted to tadpole graphs, several corona product of paths (Bilal et al., 2020), certain Cartesian product of graphs, trees (Cichacz et al., 2022), join product of graphs (Susanto et al., 2022), corona product involving complete graphs, friendship graphs, Jahangir graphs, helm graphs (Ahmad, 2023), and generalized complete friendship graphs (Majid et al., 2023). In addition, Cichacz et al. (2022) determined the upper bound of distance irregularity strength for any graphs in general.

Moreover, Bača et al. (2018) considered a slight variation of DIL, called inclusive distance irregular labeling. The k-labeling $f : V(G) \rightarrow [1, k]$ is an *inclusive distance irregular labeling* (IDIL) if for every two distinct vertices u and v, $w(u) \neq w(v)$ with $w(u) = f(u) + \sum_{x \in N(u)} f(x)$. The *inclusive distance irregularity strength* (IDIS), denoted by $\widehat{\mathrm{dis}}(G)$, of G is the least integer k where G has an inclusive distance irregular k-labeling. Investigations of IDIS-ness of graphs are conducted for spider graphs, forests (Cichacz et al., 2021), join product of graphs (Susanto et al., 2022), book graphs (Wahyu et al., 2023), and n-ary tree (Wijaya et al., 2024). The upper bound of IDIS of graphs also has been determined by Cichacz et al. (2021).

Motivated by the notion of DIL and IDIL, several authors consider more generalizations or adding restrictions towards the labeling. There are studies of DIL and IDIL where the weights of vertices has additional sums (Bong et al., 2020; Utami et al., 2020; Wijayanti et al., 2023; Susanto et al., 2024). Several authors also considered a local version of IDIL (Sugeng et al., 2021; Hadiputra et al., 2023).

In this paper, we investigate the DIS and IDIS of complement graphs and in particular split graphs. Since the parameters $\mathrm{dis}(G)$ and $\widehat{\mathrm{dis}}(G)$ are closely related, we are able to provide conditions when a graph has $\mathrm{dis}(G) = \widehat{\mathrm{dis}}(G)$. Moreover, we present a new upper bound for DIS and IDIS of any given graph. In addition, we determine the DIS and IDIS of several families of split graphs. To solve a certain problem in later sections, we will also determine the vertex H-irregularity strength of cycles.

2. EXPERIMENTAL SECTION

2.1 Method

The study starts by conducting literature review on irregularity strength of graphs and its variants. We take note on important known results which are relatable to our study. These known results are written in Subsection 3.1.

Further, we continue to obtain new theorems regarding DIS and IDIS of the complement of a graph. The proofs of the theorems must be done systematical and rigorous manner. This part is shown in Subsection 3.2.

Then, the study is focused on several families of split graphs. For each family of split graph, we introduce them using the formal definition of a graph, followed by its newly determined DIS and IDIS. Subsection 3.3 covers the discussion of the split graphs.

3. RESULTS AND DISCUSSION

3.1 Preliminary and Auxiliary Results

In this section, we start by discussing preliminary results of DIL and IDIL of graphs. Cichacz et al. (2022) established a sufficient condition when a graph G has $\mathrm{dis}(G) = \infty$.

Theorem 1 *Cichacz et al. (2022), For a graph G, if there exist two distinct vertices $u, v \in V(G)$ such that $N_G(u) = N_G(v)$, then $dis(G) = \infty$.*

In addition, if the graph G has no two vertices x and y which satisfy $N_G(u) \neq N_G(v)$, then the graph admits a DIL.

Theorem 2 *Cichacz et al. (2022) If $N_G(u) \neq N_G(v)$ for every two distinct vertices $u, v \in V(G)$, then $dis(G) \leq \Delta(G)(|V(G)| - 1) + 1$.*

Moreover, a characterization of graphs having $\widehat{\mathrm{dis}}(G) = \infty$ is determined by Bača et al. (2018).

Theorem 3 *Bača et al. (2018) For a graph G holds $\widehat{dis}(G) = \infty$ if and only if there exist two distinct vertices $u, v \in V(G)$ such that $N_G[u] = N_G[v]$. In particular, if a graph has the property that $N_G[u] \neq N_G[v]$ for every two distinct vertices $u, v \in V(G)$, then $\widehat{dis}(G) \leq 2^{|V(G)|}$.*

An improved upper bound for IDIS of any graphs are also provided by Cichacz et al. (2021).

Theorem 4 *Cichacz et al. (2021) Let G be a graph such that $N_G[u] \neq N_G[v]$ for any distinct vertices $u, v \in V(G)$. Then $\widehat{dis}(G) \leq (\Delta(G) + 1)(|V(G)| - 1) + 1$.*

Next, we will discuss other kind of labelings which are still relevant in this study. Faudree et al. (1989) determined the irregularity strength of cycles C_n. The result is presented below.

Theorem 5 *Faudree et al. (1989) Let $n \geq 3$ be an integer. It follows that*

$$s(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{for } n \equiv 1 \ (\text{mod } 4), \\ \lceil \frac{n}{2} \rceil + 1, & \text{otherwise}. \end{cases} \tag{2}$$

In order for them to prove it, they present a labeling $f : E(C_n) \rightarrow \mathbb{Z}$ based on the condition of n. Let $V(C_n) = \{v_1, \ldots, v_n\}$ and $e_i = v_{i-1}v_i$ where the index is taken modulo n. If $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, the labeling f is defined by

$$f(e_i) = \begin{cases} i, & \text{for } i \in [1, \lceil \frac{n+1}{2} \rceil], \\ 2\lceil \frac{n-i}{2} \rceil + 1, & \text{for } i \in [\lceil \frac{n+3}{2} \rceil, n]. \end{cases} \tag{3}$$

Meanwhile, if $n \equiv 2 \pmod 4$ then f is defined where

$$f(e_i) = \begin{cases} i, & \text{for } i \in [1, \frac{n}{2} - 1], \\ \frac{n}{2} + 1, & \text{for } i \in \{\frac{n}{2}, \frac{n}{2} + 1\}, \\ 2\lceil \frac{n-i}{2} \rceil + 1, & \text{for } i \in [\frac{n}{2} + 2, n]. \end{cases} \tag{4}$$

Lastly, if $n \equiv 3 \pmod 4$ then f is defined such that

$$f(e_i) = \begin{cases} i, & \text{for } i \in [1, \frac{n-1}{2}], \\ n - i + 2, & \text{for } i \in \{\frac{n+1}{2}, \frac{n+3}{2}\}, \\ 2\lceil \frac{n-i+1}{2} \rceil + 1, & \text{for } i \in [\frac{n+5}{2}, n]. \end{cases} \tag{5}$$

Now, let f be an irregularity labeling of C_n. It is provided in Ahmad et al. (2014) that one is able to have the edge irregularity labeling of C_n by simply 'shifting' the labels of the edges to the vertices. In formal way, we are able to obtain $g : V(C_n) \to \mathbb{Z}$ defined by $g(v_i) = f(e_i)$ which implies g is an edge irregularity labeling of C_n.

Corollary 1 *Let $n \geq 3$ be an integer. It holds that*

$$es(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{for } n \equiv 1 \ (mod\ 4), \\ \lceil \frac{n}{2} \rceil + 1, & \text{otherwise}. \end{cases} \tag{6}$$

In the process of determining Theorem 14 in Section 4, we found an auxiliary result that can be independent on its own. Therefore as the end of this section, we would like to provide a new result about vertex path-irregularity strengthness regarding cycles.

Theorem 6 *Let $n \geq 2$ be an integer. Then, it follows that $vhs(C_{2n}, P_n) = 3$.*

Proof. To show that $vhs(C_{2n}, P_n) \geq 3$, for $i \in [1, 2n]$ let $P_n^{(i)}$ be a subgraph of C_{2n} where $V(P_n^{(i)}) = \{v_i, \ldots, v_{n-i+1}\}$ and $E(P_n^{(i)}) = \{v_i v_{i+1}, \ldots, v_{n-i} v_{n-i+1}\}$ where the index is taken modulo n. Let $f : V(C_n) \to \mathbb{Z}$ be a vertex P_n-irregularity labeling of C_n and $wt_f : \{P_n^{(i)} \mid i \in [1, 2n]\} \to \mathbb{Z}$ be a weight map induced by f. It is obvious that $wt_f(P_n^{(i)}) \geq n$ for any labeling f. Since the weights $wt_f(P_n^{(i)})$ must be distinct, then there exists $k \in [1, 2n]$ where $wt_f(P_n^{(k)}) \geq 3n - 1$. This implies there exists $l \in [1, 2n]$ where $f(v_l) \geq 3$. Therefore, $vhs(C_{2n}, P_n) \geq 3$.

Next, we will show that $vhs(C_{2n}, P_n) \leq 3$. Let $f : V(C_{2n}) \to \{1, 2, 3\}$ where

$$f(v_i) = \begin{cases} 3, & \text{for } i \in [\lfloor \frac{n}{2} \rfloor + 1, 2n - \lfloor \frac{n}{2} \rfloor - 1], \\ 2, & \text{for } i = 2n - \lfloor \frac{n}{2} \rfloor, \\ 1, & \text{otherwise}. \end{cases} \tag{7}$$

Now, the induced weights $wt_f : \{P_n^{(i)} \mid i \in [1, 2n]\} \to \mathbb{Z}$ from f is obtained where

$$wt_f(P_n^{(i)}) = \begin{cases} n + 2i + 2, & \text{for } i \in [1, \lceil \frac{n}{2} \rceil], \\ 3n - 2(i - \lceil \frac{n}{2} \rceil) + 1, & \text{for } i \in [\lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + n], \\ 2(i - \lceil \frac{n}{2} \rceil - 1) - n, & \text{for } i \in [\lceil \frac{n}{2} \rceil + n + 1, 2n]. \end{cases} \tag{8}$$

It is a routine to check that all the weights are distinct. Since the largest label of f is 3, then $vhs(C_{2n}, P_n) \leq 3$ which implies $vhs(C_{2n}, P_n) = 3$.

3.2 Complement of graphs

Let G be a finite, simple and undirected graph. A complement of a graph $\overline{G}$ is a graph obtained by the vertices $V(\overline{G}) = V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Theorem 7 *Let G be a graph. It follows that $\widehat{dis}(\overline{G}) = dis(G)$.*

Proof.Let G be a graph such that $\text{dis}(G) < \infty$. Let $f : V(G) \to \mathbb{Z}$ be a DIL of G which induces the weight map $wt_f : V(G) \to \mathbb{Z}$ for some positive integer k where $wt_f(v) = \sum_{v \in N(v)} f(v)$. We will show that f is also an IDIL of $\overline{G}$. Let $wt'_f(v) = \sum_{v \in N[v]} f(v)$ be a weight function induced by f. For any two distinct vertices u and v, it follows that

$$\begin{aligned} wt_f(v) &= wt_f(u), \\ \sum_{x \in N_G(v)} f(x) &= \sum_{x \in N_G(u)} f(x), \\ \sum_{x \in V(G)} f(x) - \sum_{x \in N_G(v)} f(x) &= \sum_{x \in V(G)} f(x) - \sum_{x \in N_G(u)} f(x), \\ \sum_{x \in N_{\overline{G}}[v]} f(x) &= \sum_{x \in N_{\overline{G}}[u]} f(x), \\ wt'_f(v) &= wt'_f(u). \end{aligned} \tag{9}$$

Therefore, f is also an IDIL of G. This implies $\widehat{\text{dis}}(\overline{G}) \leq \text{dis}(G)$. The proof to show $\text{dis}(G) \leq \widehat{\text{dis}}(\overline{G})$ is done similarly.

Now, let $\text{dis}(G) = \infty$. It follows that $\widehat{\text{dis}}(\overline{G}) = \infty$. If not, say $\widehat{\text{dis}}(\overline{G}) < \infty$, then we can follow previous steps to conclude that $\text{dis}(G) < \infty$ which is a contradiction.

Now, by combining Theorem 7 and Theorem 2, we are able to show another upper bound for IDIS of a graph if the neighbors $N_G[v]$ of every vertex v in G are pairwise distinct. This would be useful if we consider graphs with high minimum degree.

Theorem 8 *Let G be a graph with $N_G[u] \neq N_G[v]$ for every two distinct vertices $u, v \in V(G)$. It holds that $\widehat{dis}(G) \leq (|V(G)| - \delta(G) - 1)(|V(G)| - 1) + 1$.*

Proof. Consider $\overline{G}$, that is a complement of G. Since $N_G[u] \neq N_G[v]$ for every two distinct vertices u and v in G, then $N_{\overline{G}}(u) \neq N_{\overline{G}}(v)$ for every two distinct vertices $u, v \in V(\overline{G})$. By Theorem 2, then $\text{dis}(\overline{G}) \leq \Delta(\overline{G})(|V(\overline{G})| - 1) + 1$. Note that $|V(\overline{G})| = |V(G)|$ and $\Delta(\overline{G}) = |V(G)| - \delta(G) - 1$. Therefore, $\widehat{\text{dis}}(G) \leq (|V(G)| - \delta(G) - 1)(|V(G)| - 1) + 1$.

By similar approach, we are able to get another upper bound for DIS of a graph satisfying the condition.

Theorem 9 *Let G be a graph with $N_G(u) \neq N_G(v)$ for every two distinct vertices $u, v \in V(G)$. It holds that $dis(G) \leq (|V(G)| - \delta(G))(|V(G)| - 1) + 1$.*

Proof. The proof is similar to Theorem 8, by combining Theorem 7 and Theorem 4.

A graph G is said to be *self-complementary* if $\overline{G} = G$. Using this notion, we have a sufficient condition for graphs G to have $\widehat{\text{dis}}(G) = \text{dis}(G)$.

Corollary 2 *Let G be a self-complementary graph. Then $\widehat{dis}(G) = dis(G)$.*

We will see examples for the Corollary 2 in the end of the next section.

3.3 Split Graphs

Split graph is one of the interesting class graph which is closed under complementation, since the complement of a split graph is also a split graph. In this section, we will determine the non-inclusive and inclusive distance vertex irregularity strength of split graph. For some integers $n \geq 3$ and $m \geq 1$, a split graph G is a graph where $V(G) = V_1 \cup V_2$ such that V_1 induces a clique with $|V_1| = n$ and V_2 induces an independent set where $|V_2| = m$. In this study, we restrict for the degrees of V_2 to be at most $n - 1$, since the vertex with the degree n in V_2 can be considered as a part of the clique, which implies the graph would be isomorphic to some other split graph G' where $V(G') = V_1' \cup V_2'$, $|V_1'| = n + 1$, $|V_2'| = m - 1$.

First, we will see how low can the parameters $\text{dis}(G)$ and $\widehat{\text{dis}}(G)$ be for split graphs G. Let $n \geq 2$ and $m \geq 1$ be integers. Let $T_{n,m}$ be a *monotonous* split graph with the vertex set $V(T_{n,m}) = V_1 \cup V_2$ where

$$\begin{aligned} V_1 &= \{v_i \mid i \in [1, n+m]\}, \\ V_2 &= \{u_i \mid i \in [1, n]\}, \end{aligned} \tag{10}$$

and the edge set

$$\begin{aligned} E(T_{n,m}) = \{v_i v_j \mid i, j \in [1, n+m],\ i \neq j\} \\ \cup \{v_i u_j \mid i \in [1, n],\ j \in [1, i]\}. \end{aligned} \tag{11}$$

We present an example of $T_{3,2}$ in Figure 1

Theorem 10 *Let $n \geq 2$ and $m \geq 1$ be integers. Then, it holds that*

$$dis(T_{n,m}) = \begin{cases} \infty, & \text{if } m = 1, \\ m, & \text{if } m \geq 2. \end{cases} \tag{12}$$

In addition, it follows that

$$\widehat{dis}(T_{n,m}) = \begin{cases} 2, & \text{if } m = 1, \\ \infty, & \text{if } m \geq 2. \end{cases} \tag{13}$$

Proof. First, we will show that (12) holds. If $m = 1$, then $N_{T_{n,1}}(v_{n+1}) = N_{T_{n,1}}(u_n)$. This implies $\text{dis}(T_{n,1}) = \infty$ due to Theorem 1. If $m \geq 2$, then $N_{T_{n,m}}[v_i] = N_{T_{n,m}}[v_j]$ for every $i, j \in [n+1, m]$. This implies that the labels of v_i where $i \in [n+1, n+m]$ must be distinct. Hence, $\text{dis}(T_{n,m}) \geq m$.

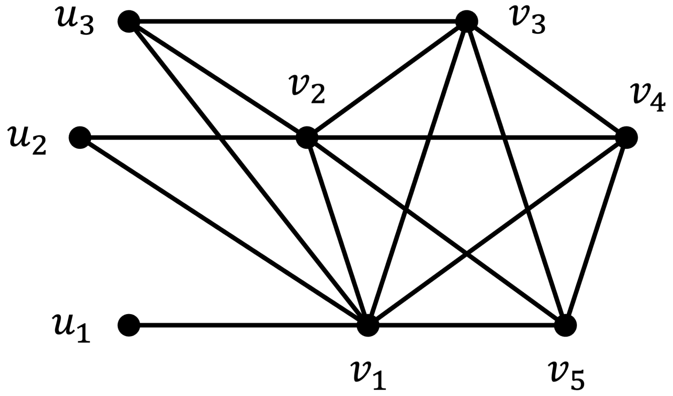

Figure 1. The Split Graph $T_{3,2}$

To show $\text{dis}(T_{n,m}) \leq m$, define a labeling $f : V(T_{n,m}) \to [1, 2]$ where

$$\begin{aligned} f(v_i) &= \begin{cases} 1, & \text{for } i \in [1, n], \\ i - n, & \text{for } i \in [n+1, n+m], \end{cases} \\ f(u_i) &= 1 \qquad \text{for } i \in [1, n]. \end{aligned} \tag{14}$$

Observe that the weights $wt_f : V(T_{n,m}) \to \mathbb{Z}$ induced by f are

$$\begin{aligned} wt_f(v_i) &= \begin{cases} n + i - 1 + \frac{m(m+1)}{2}, & \text{for } i \in [1, n], \\ 2n - i + \frac{m(m+1)}{2}, & \text{for } i \in [n+1, n+m], \end{cases} \\ wt_f(u_i) &= n - i + 1 \qquad \text{for } i \in [1, n]. \end{aligned} \tag{15}$$

It is not hard to show that all the weights are distinct. This implies $\text{dis}(T_{n,m}) = m$.

Next, we will show (13) also holds. If $m \geq 2$, then $N_{T_{n,m}}[v_{n+1}] = N_{T_{n,m}}[v_{n+2}]$. Hence, $\widehat{\text{dis}}(T_{n,m}) = \infty$ due to Theorem 3. If $m = 1$, let $f : V(T_{n,1}) \to [1, 2]$ be a labeling such that

$$\begin{aligned} f(v_i) &= \begin{cases} 1, & \text{for } i \in [1, n], \\ 2, & \text{for } i = n + 1, \end{cases} \\ f(u_i) &= 1 \qquad \text{for } i \in [1, n]. \end{aligned} \tag{16}$$

Observe that the weights $wt_f : V(T_{n,1}) \to \mathbb{Z}$ induced by f are

$$\begin{aligned} wt_f(v_i) &= \begin{cases} n + i + 2, & \text{for } i \in [1, n], \\ n + 2, & \text{for } i = n + 1, \end{cases} \\ wt_f(u_i) &= n - i + 2 \qquad \text{for } i \in [1, n]. \end{aligned} \tag{17}$$

Again, it is evident that all the weights are distinct. Therefore, $\widehat{\text{dis}}(T_{n,1}) = 2$.

From the preceding theorem, it is evident that $\text{dis}(T_{n,2}) = 2$. Hence, the DIS and IDIS of a split graph can be as low as 2.

Now, we shift our concern to see a special case of product of graphs which is also a split graph. The corona product of two graphs G_1 and G_2, denoted by $G_1 \odot G_2$ is defined as the graph G obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining by a line the i-th point of G_1 to every point in the i-th copy of G_2 (Frucht and Harary, 1970). The special case of this product that yields a split graph is when $G_1 = K_n$ and $G_2 = \overline{K_m}$ for some integers $n \geq 3$ and $m \geq 1$.

Theorem 11 *Let $n \geq 3$ and $m \geq 1$ be integers. Then*

$$dis(K_n \odot \overline{K_m}) = \begin{cases} n, & \text{for } m = 1, \\ \infty, & \text{for } m \geq 2. \end{cases} \tag{18}$$

Furthermore, it also holds that

$$\widehat{dis}(K_n \odot \overline{K_m}) = \begin{cases} n, & \text{for } m = 1, \\ \lceil \frac{nm+1}{2} \rceil, & \text{for } m \geq 2. \end{cases} \tag{19}$$

Proof. Define the graph $K_n \odot \overline{K_m}$ as a graph with the vertex set

$$\begin{aligned} V(K_n \odot \overline{K_m}) = &\{v_i \mid i \in [1, n]\} \\ &\cup \{u_{i,j} \mid i \in [1, n],\ j \in [1, m]\}. \end{aligned} \tag{20}$$

and the edge set

$$\begin{aligned} E(K_n \odot \overline{K_m}) = &\{v_i v_k \mid i, k \in [1, n],\ i \neq k\} \\ &\cup \{v_i u_{i,j} \mid i \in [1, n],\ j \in [1, m]\}. \end{aligned} \tag{21}$$

First, we will show (18). If $m \geq 2$, then $N_{K_n \odot \overline{K_m}}(u_{1,1}) = N_{K_n \odot \overline{K_m}}(u_{1,2})$. This implies $\mathrm{dis}(K_n \odot \overline{K_m}) = \infty$ due to Theorem 1. Now, let $m = 1$. Observe that in any distance irregular labeling $f : V(K_n \odot \overline{K_1}) \to \mathbb{Z}$ which induces a weight map $wt_f : V(K_n \odot \overline{K_1}) \to \mathbb{Z}$, it holds that $wt_f(u_{i,1}) = f(v_i)$. Since these weights must be distinct, then there exists $k \in [1, n]$ such that $f(v_k) \geq n$. This immediately implies $\mathrm{dis}(K_n \odot \overline{K_1}) \geq n$.

To show $\mathrm{dis}(K_n \odot \overline{K_1}) \leq n$, let $f : V(K_n \odot \overline{K_1}) \to [1, n]$ be a map where

$$\begin{aligned} f(v_i) &= i, && \text{for } i \in [1, n], \\ f(u_{i,1}) &= n, && \text{for } i \in [1, n]. \end{aligned} \tag{22}$$

The labeling f induces a weight function $wt_f : V(K_n \odot \overline{K_1}) \to \mathbb{Z}$ such that

$$\begin{aligned} wt_f(v_i) &= n - i + \frac{n(n+1)}{2}, && \text{for } i \in [1, n], \\ wt_f(u_{i,1}) &= i, && \text{for } i \in [1, n]. \end{aligned} \tag{23}$$

It is not hard to check that this weights are distinct. Therefore, $\mathrm{dis}(K_n \odot \overline{K_1}) \leq n$. This shows (18).

Now, we will show (19). Let $m = 1$. It is evident that in any inclusive distance irregular labeling $f : V(K_n \odot \overline{K_1}) \to \mathbb{Z}$ which induces a weight function $wt_f : V(K_n \odot \overline{K_1}) \to \mathbb{Z}$, we have $wt_f(v_i) = f(u_{i,1}) + \sum_{k=1}^{n} f(v_k)$. Since these weights are distinct and $\sum_{k=1}^{n} f(v_k)$ is constant, then there exists $j \in [1, n]$ such that $f(u_{j,1}) \geq n$. Consequently, $\widehat{\mathrm{dis}}(K_n \odot \overline{K_1}) \geq n$.

To show $\widehat{\mathrm{dis}}(K_n \odot \overline{K_1}) \leq n$, consider a map $f : V(K_n \odot \overline{K_1}) \to [1, n]$ where

$$\begin{aligned} f(v_i) &= 1, && \text{for } i \in [1, n], \\ f(u_{i,1}) &= i, && \text{for } i \in [1, n]. \end{aligned} \tag{24}$$

The labeling f induces a weight function $wt_f : V(K_n \odot \overline{K_1}) \to \mathbb{Z}$ such that

$$\begin{aligned} wt_f(v_i) &= i + \frac{n(n+1)}{2}, && \text{for } i \in [1, n], \\ wt_f(u_{i,1}) &= i + 1, && \text{for } i \in [1, n]. \end{aligned} \tag{25}$$

It is obvious that this weights are distinct. Then, $\widehat{\mathrm{dis}}(K_n \odot \overline{K_1}) \leq n$ which implies $\widehat{\mathrm{dis}}(K_n \odot \overline{K_1}) = n$.

Next, let $m \geq 2$. Let $f : V(K_n \odot \overline{K_m}) \to \mathbb{Z}$ be an inclusive distance irregular labeling of $K_n \odot \overline{K_m}$ that induces a weight map $wt_f : V(K_n \odot \overline{K_m}) \to \mathbb{Z}$. For $i \in [1, n]$ and $j \in [1, m]$, observe that the weights $wt_f(u_{i,j})$ must be distinct. Hence, there exists $k \in [1, n]$ and $l \in [1, m]$ where $wt_f(u_{k,l}) \geq nm + 1$. This implies that $f(v_k) \geq \lceil \frac{nm+1}{2} \rceil$ or $f(u_{k,l}) \geq \lceil \frac{nm+1}{2} \rceil$. In any case, it follows that $\widehat{\mathrm{dis}}(K_n \odot \overline{K_m}) \geq \lceil \frac{nm+1}{2} \rceil$.

To show $\widehat{\mathrm{dis}}(K_n \odot \overline{K_m}) \leq \lceil \frac{nm+1}{2} \rceil$, we will divide the problem in cases based on the parity of m.

Case 1. Let m be an even number. Define a labeling $f : V(K_n \odot \overline{K_m}) \to [1, \lceil \frac{nm+1}{2} \rceil]$ such that

$$\begin{aligned} f(v_i) &= \begin{cases} 1, & \text{for } i = 1, \\ \frac{im}{2}, & \text{for } i \in [2, n], \end{cases} \\ f(u_{i,j}) &= \begin{cases} j, & \text{for } i = 1, j \in [1, m], \\ \frac{m}{2}(i-2) + j + 1, & \text{for } i \in [2, n], j \in [1, m]. \end{cases} \end{aligned} \tag{26}$$

This map f induces a weight map $wt_f : V(K_n \odot \overline{K_m}) \to \mathbb{Z}$ where

$$\begin{aligned} wt_f(v_i) &= \begin{cases} \frac{m}{2}(\frac{n(n+1)+(m-1)}{2}) + 1, & \text{for } i = 1, \\ \frac{m}{2}(\frac{n(n+1)+(m-1)}{2} + m(i-2) + 2) + 1, & \text{for } i \in [2, n], \end{cases} \\ wt_f(u_{i,j}) &= m(i-1) + j + 1, \quad \text{for } i \in [1, n], j \in [1, m]. \end{aligned} \tag{27}$$

We can use induction to show that $\max wt_f(u_{i,j}) < \min wt_f(v_i)$. This implies that these weights are distinct.

Case 2. Let m be an odd number. Let $f : V(K_n \odot \overline{K_m}) \to [1, \lceil \frac{nm+1}{2} \rceil]$ where

$$\begin{aligned} f(v_i) &= \begin{cases} 1, & \text{for } i = 1, \\ m, & \text{for } i = 2, \\ m + \frac{(i-2)(m-1)}{2} + \lceil \frac{i-1}{2} \rceil, & \text{for } i \in [3, n], \end{cases} \\ f(u_{i,j}) &= \begin{cases} j, & \text{for } i = 1, j \in [1, m], \\ j + 1, & \text{for } i = 2, j \in [1, m], \\ \frac{(i-2)(m+1)}{2} + j + 1 - \lceil \frac{i-1}{2} \rceil, & \text{for } i \in [3, n], j \in [1, m]. \end{cases} \end{aligned} \tag{28}$$

Let $\theta = \frac{1}{2}\left(\frac{(n-2)(n-1)(m-1)}{2} + \lceil\frac{n-2}{2}\rceil\lceil\frac{n}{2}\rceil + \lceil\frac{n-1}{2}\rceil\lceil\frac{n+1}{2}\rceil\right)$. Then, the labeling f induces weight function $wt_f : V(K_n \odot \overline{K_m}) \rightarrow \mathbb{Z}$ such that

$$wt_f(v_i) = \begin{cases} (\frac{m+1}{2} + n - 1)m + \theta, & \text{for } i = 1, \\ (\frac{m+1}{2} + n)m + \theta, & \text{for } i = 2, \\ (\frac{(i-2)(m+1)}{2} - \lceil\frac{i-1}{2}\rceil + \frac{m+1}{2} + n)m + \theta, & \text{for } i \in [3, n], \end{cases} \quad (29)$$
$$wt_f(u_{i,j}) = m(i-1) + j + 1, \qquad \text{for } i \in [1, n], j \in [1, m].$$

Similarly, it is a routine to check that these weights are distinct.

In both cases, it is evident that the largest label of f is $\lceil\frac{nm+1}{2}\rceil$. Therefore, $\widehat{\mathrm{dis}}(K_n \odot \overline{K_m}) \le \lceil\frac{nm+1}{2}\rceil$ which implies $\widehat{\mathrm{dis}}(K_n \odot \overline{K_m}) = \lceil\frac{nm+1}{2}\rceil$. This shows the theorem.

Previously, we have seen a split graph where all vertices in the independent set are pendants. Now, we consider a split graph where half of the independent set are pendants and the other half is not. Let $n \ge 2$ be an integer. Let H_{2n} be an *half-pendant* split graph with a vertex set $V(H_{2n}) = V_1 \cup V_2$ where

$$\begin{aligned} V_1 &= \{v_i \mid i \in [1, 2n]\}, \\ V_2 &= \{u_i \mid i \in [1, 2n]\}, \end{aligned} \quad (30)$$

and the edge set

$$\begin{aligned} E(H_{2n}) = \{v_i v_j \mid i, j \in [1, 2n], i \ne j\} \cup \{v_i u_i \mid i \in [1, n]]\} \\ \cup \{v_i u_j \mid i \in [1, 2n], j \in [n+1, 2n], i \ne j\}. \end{aligned} \quad (31)$$

An example of H_4 is provided in Figure 2

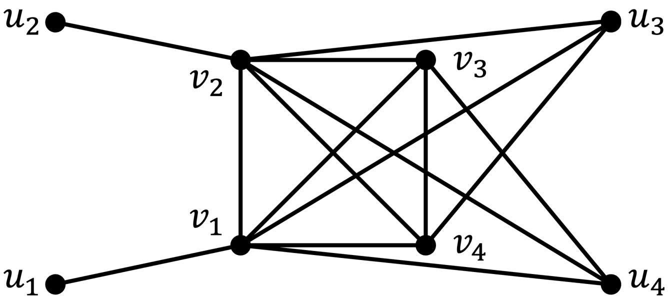

Figure 2. The Split Graph H_4

Here, we are able to show a first kind of split graph H_{2n} where $\mathrm{dis}(H_{2n}) = \widehat{\mathrm{dis}}(H_{2n})$.

Theorem 12 *Let $n \ge 2$ be an integer. It holds that*

$$dis(H_{2n}) = \widehat{dis}(H_{2n}) = n \quad (32)$$

Proof. First, we will show that $\mathrm{dis}(H_{2n}) = n$. Let $f : V(H_{2n}) \rightarrow \mathbb{Z}$ be a distance irregular labeling of H_{2n} which induces a weight function $wt_f : V(H_{2n}) \rightarrow \mathbb{Z}$. Observe that $wt_f(u_i) = f(v_i)$ for $i \in [1, n]$. Since the weights $wt_f(u_i)$ must be distinct for $i \in [1, n]$, then there exists $k \in [1, n]$ where $f(v_i) \ge k$. This implies $\mathrm{dis}(H_{2n}) \ge n$.

To show that $\mathrm{dis}(H_{2n}) \le n$, consider the labeling $f : V(H_{2n}) \rightarrow [1, n]$ where

$$\begin{aligned} f(v_i) &= \begin{cases} i, & \text{for } i \in [1, n], \\ i - n, & \text{for } i \in [n+1, 2n], \end{cases} \\ f(u_i) &= \begin{cases} 1, & \text{for } i \in [1, n], \\ n, & \text{for } i \in [n+1, 2n]. \end{cases} \end{aligned} \quad (33)$$

Observe that the weights $wt_f : V(H_{2n}) \rightarrow \mathbb{Z}$ induced by f are

$$\begin{aligned} wt_f(v_i) &= \begin{cases} n(2n+1) - i + 1, & \text{for } i \in [1, n], \\ n(2n+1) - i, & \text{for } i \in [n+1, 2n], \end{cases} \\ wt_f(u_i) &= \begin{cases} i, & \text{for } i \in [1, n], \\ n(n+1) + n - i, & \text{for } i \in [n+1, 2n]. \end{cases} \end{aligned} \quad (34)$$

It is not hard to see that the weights are distinct. Therefore, $\mathrm{dis}(H_{2n}) \le n$ which implies $\mathrm{dis}(H_{2n}) = n$.

that $\widehat{\mathrm{dis}}(H_{2n}) \le n$. Let $f : V(H_{2n}) \rightarrow [1, n]$ be a labeling where

$$\begin{aligned} f(v_i) &= 1, \qquad \text{for } i \in [1, 2n], \\ f(u_i) &= \begin{cases} i, & \text{for } i \in [1, n], \\ i - n, & \text{for } i \in [n+1, 2n]. \end{cases} \end{aligned} \quad (35)$$

Then, the weights $wt_f : V(H_{2n}) \rightarrow \mathbb{Z}$ induced by f are

$$\begin{aligned} wt_f(v_i) &= \begin{cases} \frac{n}{2}(n+5) + i, & \text{for } i \in [1, n], \\ \frac{n}{2}(n+7) - i, & \text{for } i \in [n+1, 2n], \end{cases} \\ wt_f(u_i) &= \begin{cases} i + 1, & \text{for } i \in [1, n], \\ n + i - 1, & \text{for } i \in [n+1, 2n]. \end{cases} \end{aligned} \quad (36)$$

Likewise, it is not hard to see that the weights are distinct. It follows that $\widehat{\mathrm{dis}}(H_{2n}) \le n$ which implies $\widehat{\mathrm{dis}}(H_{2n}) = n$. This shows the theorem.

Next, we will see a little connection of non-inclusive and inclusive distance irregular labeling with edge irregular labeling. Let S_n be a sun graph with a vertex set $V(S_n) = V_1 \cup V_2$ where

$$\begin{aligned} V_1 &= \{v_i \mid i \in [1, n]\}, \\ V_2 &= \{u_i \mid i \in [1, n]\}, \end{aligned} \quad (37)$$

and an edge set

$$E(S_n) = \{v_i v_j \mid i, j \in [1, n], i \ne j\} \cup \{v_i u_i, v_{i+1} u_i \mid i \in [1, n]\} \quad (38)$$

where the index i is taken modulo n. It is evident that a sun graph is another special case of a split graph.

Theorem 13 *Let $n \ge 3$ be an integer and C_n be a cycle on n vertices. It follows that*

$$dis(S_n) = \widehat{dis}(S_n) = es(C_n) = \begin{cases} \lceil\frac{n}{2}\rceil, & \textit{for } n \equiv 1 \pmod 4, \\ \lceil\frac{n}{2}\rceil + 1, & \textit{otherwise}. \end{cases} \quad (39)$$

Proof. First, we will show that $\mathrm{dis}(S_n) = es(C_n)$. Let $f : V(S_n) \to \mathbb{Z}$ be a distance irregular k-labeling of S_n. Let $V(C_n) = \{x_1, \ldots, x_n\}$ and $E(C_n) = \{x_i x_{i+1} \mid i \in [1, n]\}$ where the index i is taken modulo n. Let $g : V(C_n) \to [1, k]$ where $g(x_i) = f(v_i)$ for $i \in [1, n]$. Let $wt_f : V(S_n) \to \mathbb{Z}$ be the weight map induced by f and $wt_g : E(C_n) \to \mathbb{Z}$ be the weight function induced by g. Since the weights $wt_f(u_i) = f(v_i) + f(v_{i+1})$ in S_n must be distinct, then $g(x_i) + g(x_{i+1}) = wt_g(x_i x_{i+1})$ also must be distinct. This implies g is an edge irregular k-labeling of C_n. Therefore, $es(C_n) \le \mathrm{dis}(S_n)$.

To show that $\mathrm{dis}(S_n) \le es(C_n)$, let $es(C_n) = k$. Let $g : V(C_n) \to [1, k]$ be an edge irregular k-labeling of C_n and let $wt_g : E(C_n) \to \mathbb{Z}$ be a weight map induced by g. Define a labeling $f : V(S_n) \to [1, k]$ where

$$\begin{aligned} f(v_i) &= g(x_i), && \text{for } i \in [1, n], \\ f(u_i) &= g(x_{i-\lceil \frac{n}{2} \rceil + 1}), && \text{for } i \in [1, n], \end{aligned} \tag{40}$$

where the index is taken modulo n. Let $\theta = \sum_{j=1}^{n} g(x_j)$ and let $wt_f : V(S_n) \to \mathbb{Z}$ be a weight function induced by f. Then, we have

$$\begin{aligned} wt_f(v_i) &= \theta - g(x_i) + wt_g(x_{i-\lceil \frac{n}{2} \rceil} x_{i-\lceil \frac{n}{2} \rceil + 1}), && \text{for } i \in [1, n], \\ wt_f(u_i) &= wt_g(x_i x_{i+1}), && \text{for } i \in [1, n]. \end{aligned} \tag{41}$$

It is a routine to check that all the weights are distinct. This shows $\mathrm{dis}(S_n) \le k$ which implies $\mathrm{dis}(S_n) = es(C_n)$.

Next, we want to show that $\widehat{\mathrm{dis}}(S_n) = es(C_n)$. Let $f : V(S_n) \to [1, k]$ be an inclusive distance irregular k-labeling of S_n. Let $g : V(C_n) \to [1, k]$ where $g(x_i) = f(u_i)$ for $i \in [1, n]$. Let $wt_f : V(S_n) \to \mathbb{Z}$ be the weight function induced by f and $wt_g : E(C_n) \to \mathbb{Z}$ be the weight map induced by g. Let $\theta = \sum_{j=1}^{n} f(v_i)$. Since the weights $wt_f(v_i) = \theta + f(u_{i-1}) + f(u_i)$ in S_n must be distinct, then $g(x_{i-1}) + g(x_i) = wt_g(x_{i-1} x_i)$ also must be distinct. This implies g is an edge irregular k-labeling of C_n. Therefore, $es(C_n) \le \widehat{\mathrm{dis}}(S_n)$.

Lastly, we will show that $\widehat{\mathrm{dis}}(S_n) \le es(C_n)$. Let $es(C_n) = k$. Let $g : V(C_n) \to [1, k]$ be an edge irregular k-labeling of C_n and let $wt_g : \{P_n^{(i)} \mid i \in [1, 2n]\} \to \mathbb{Z}$ be a weight map induced by g. Consider a map $f : V(S_n) \to [1, k]$ such that

$$\begin{aligned} f(v_i) &= g(x_i), && \text{for } i \in [1, n], \\ f(u_i) &= g(x_{n-i+1}), && \text{for } i \in [1, n]. \end{aligned} \tag{42}$$

Let $wt_f : V(S_n) \to \mathbb{Z}$ be a weight map induced by f and let $\theta = \sum_{j=1}^{n} g(x_j)$. It follows that

$$\begin{aligned} wt_f(v_i) &= \theta + wt_g(x_{n-i} x_{n-i+1}), && \text{for } i \in [1, n], \\ wt_f(u_i) &= g(x_{n-i+1}) + wt_g(x_i x_{i+1}), && \text{for } i \in [1, n]. \end{aligned} \tag{43}$$

Again, it is a routine to check that all the weights are distinct. Hence, $\widehat{\mathrm{dis}}(S_n) \le k = es(C_n)$ and this shows the theorem.

Now as promised in the previous section, we present several examples of self-complementary split graphs. Let $n \ge 2$ be an integer. Let W_{2n} be a *twin* split graph with the vertex set $V(W_{2n}) = V_1 \cup V_2$ where

$$\begin{aligned} V_1 &= \{v_i \mid i \in [1, n]\}, \\ V_2 &= \{u_i \mid i \in [1, n]\}, \end{aligned} \tag{44}$$

and the edge set

$$\begin{aligned} E(W_{2n}) = \{&v_i v_j \mid i, j \in [1, 2n], i \neq j\} \\ &\cup \{v_i u_j \mid i, j \in [1, n] \text{ or } i, j \in [n+1, 2n]\}. \end{aligned} \tag{45}$$

It is not hard to show that the twin split graph is self-complementary.

Proposition 1 *Let $n \ge 2$ be an integer. It holds that*

$$dis(W_{2n}) = \widehat{dis}(W_{2n}) = \infty. \tag{46}$$

Proof. Observe that u_1 and u_2 are two vertices with $N_{W_{2n}}(v_1) = N_{W_{2n}}(v_2)$. By Theorem 1, then $\mathrm{dis}(G) = \infty$. Therefore, by Corollary 2, then $\widehat{\mathrm{dis}}(G) = \infty$.

Furthermore, we will provide a self-complementary split graph G with a finite $\mathrm{dis}(G)$ and $\widehat{\mathrm{dis}}(G)$. Let $n \ge 2$ be an integer. Let U_{2n} be a *uniform* split graph with the vertex set $V(W_{2n}) = V_1 \cup V_2$ where

$$\begin{aligned} V_1 &= \{v_i \mid i \in [1, n]\}, \\ V_2 &= \{u_i \mid i \in [1, n]\}, \end{aligned} \tag{47}$$

and the edge set

$$\begin{aligned} E(U_{2n}) = \{&v_i v_j \mid i, j \in [1, 2n], i \neq j\} \\ &\cup \{v_j u_i \mid i \in [1, 2n], j \in [i, i+n-1]\}, \end{aligned} \tag{48}$$

where the index j is taken modulo $2n$. We give an example of U_6 in Figure 3.

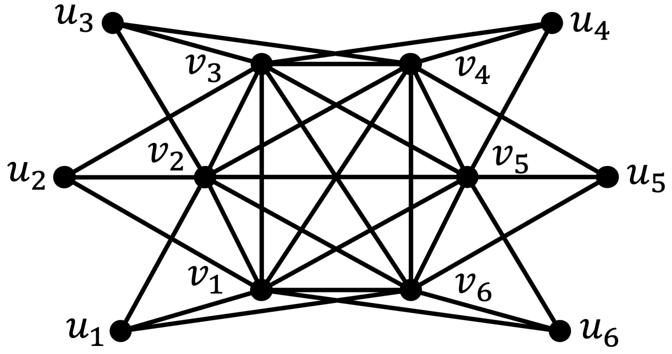

Figure 3. The Split Graph U_6

Observe that the uniform split graph is also self-complementary. In addition, if $n = 2$ then $U_4 \cong S_4$.

Theorem 14 *Let $n \geq 2$ be an integer. It follows that*

$$dis(U_{2n}) = \widehat{dis}(U_{2n}) = vhs(C_{2n}, P_n) = 3. \tag{49}$$

Proof. Since $U_4 \cong S_4$ and $vhs(C_4, P_2) = es(C_4)$, it is sufficient to only consider the case where $n \geq 3$. First, we want to show that $vhs(C_{2n}, P_n) \leq \text{dis}(U_{2n})$. Let $f : V(U_{2n}) \to [1, k]$ be a distance irregular k-labeling of U_{2n} and $wt_f : V(U_{2n}) \to \mathbb{Z}$ be a weight map induced by f. Let C_n be a graph with the vertex set $V(C_{2n}) = \{x_i \mid i \in [1, 2n]\}$ and the edge set $E(C_{2n}) = \{x_ix_{i+1} \mid i \in [1, 2n]\}$ where the index i is taken modulo $2n$. Let $P_n^{(i)}$ be a subgraph of C_{2n} where $V(P_n^{(i)}) = \{v_i, \ldots, v_{n-i+1}\}$ and $E(P_n^{(i)}) = \{v_iv_{i+1}, \ldots, v_{n-i}v_{n-i+1}\}$. Define a labeling $g : V(C_{2n}) \to [1, k]$ where $g(x_i) = f(v_i)$ for $i \in [1, 2n]$. Let $wt_g : \{P_n^{(i)} \mid i \in [1, n]\} \to \mathbb{Z}$ be a weight function induced by g. Since the weights $wt_f(u_i) = \sum_{j=i}^{i+n-1} f(v_i)$, then $\sum_{j=i}^{i+n-1} g(x_i) = wt_g(P_n^{(i)})$ also must be distinct. Therefore, g is a vertex P_n-irregular labeling of C_{2n}. This shows $vhs(C_{2n}, P_n) \leq \text{dis}(U_{2n})$.

Next, we will show that $\text{dis}(U_{2n}) \leq vhs(C_{2n}, P_n)$. Recall from Theorem 6 that $vhs(C_{2n}, P_n) = 3$. Let $g : V(C_{2n}) \to [1, 3]$ be a vertex H-irregular labeling of C_{2n} and let $wt_g : \{P_n^{(i)} \mid i \in [1, 2n]\} \to \mathbb{Z}$. Define a map $f : V(U_{2n}) \to [1, 3]$ where

$$\begin{aligned} f(v_i) &= g(x_i), && \text{for } i \in [1, 2n], \\ f(u_i) &= g(x_{2n-\lfloor \frac{n}{2} \rfloor+i}), && \text{for } i \in [1, 2n]. \end{aligned} \tag{50}$$

Let $wt_f : V(U_{2n}) \to \mathbb{Z}$ be a weight function induced by f and let $\theta = \sum_{j=1}^{n} g(x_j)$. Observe that

$$\begin{aligned} wt_f(v_i) &= \theta - g(x_i) + wt_g(P_n^{(\lceil \frac{n}{2} \rceil+1)}), && \text{for } i \in [1, 2n], \\ wt_f(u_i) &= wt_g(P_n^{(i)}), && \text{for } i \in [1, 2n]. \end{aligned} \tag{51}$$

It is a routine to check that all the weights are distinct. It follows that $\text{dis}(U_{2n}) \leq 3$ which implies $\text{dis}(U_{2n}) = vhs(C_{2n}, P_n) = 3$. By Corollary 2, then $\widehat{\text{dis}}(U_{2n}) = \text{dis}(U_{2n})$ which shows the theorem.

4. CONCLUSIONS

In Section 2, we have observed a new upper bound for DIS of any graph. That is, in Theorem 9 if $\text{dis}(G) < \infty$ then

$$\text{dis}(G) \leq (|V(G)| - \delta(G))(|V(G)| - 1) + 1. \tag{52}$$

Similarly, a new upper bound for IDIS of any graph is also presented. In Theorem 8, if $\widehat{\text{dis}}(G) < \infty$ then

$$\widehat{\text{dis}}(G) \leq (|V(G)| - \delta(G) - 1)(|V(G)| - 1) + 1. \tag{53}$$

Furthermore, in Section 3, we have determined the DIS and IDIS of several families of split graphs, namely monotonous split graphs $T_{n,m}$ (Theorem 10), special corona products $K_n \odot \overline{K_m}$ (Theorem 11), half-pendant split graphs H_{2n} (Theorem 12), sun graphs S_n (Theorem 13), twin split graphs W_{2n} (Proposition 1), and uniform split graphs U_{2n} (Theorem 14).

5. ACKNOWLEDGMENT

The authors would like to give thanks to the feedback provided by the anonymous referees.

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