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Research Paper

Relation Between Randic and Harmonic Energies of Commuting Graph for Dihedral Groups

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Abstract

Consider a finite group G with center $Z(G)$. This work examines the commuting graph Γ_G, a graph constructed from a group G whose vertices correspond precisely to the noncentral elements of the group, that is, all elements in G except those belonging to its center $Z(G)$. The graph is defined on the vertex set $G \backslash Z(G)$, where two distinct vertices v_p and v_q are joined by an edge precisely when they commute, that is, whenever $v_p v_q = v_q v_p$. The number of vertices adjacent to v_p is denoted as d_{v_p}, which is the degree of v_p. The Randic and harmonic matrices of Γ_G are defined as square matrices in which $(p, q)-$th entry are $\frac{1}{\sqrt{d_{v_p} \cdot d_{v_q}}}$ and $\frac{2}{d_{v_p}+d_{v_q}}$ if v_p and v_q are adjacents, respectively; otherwise, it is zero. Randic energy is the sum of the absolute eigenvalues of the Randic matrix whereas harmonic energy is the sum of the absolute eigenvalues of the harmonic matrix. In this paper, we compare the Randic and harmonic energies of the commuting graph for non-abelian dihedral group of order $2n$, D_{2n}.

Keywords

Randic Matrix, Harmonic Matrix, Energy of a Graph, Commuting Graph, Dihedral Group

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1. INTRODUCTION

The graph energy concept was introduced by Gutman (1978). The original framework considers the eigenvalues of the adjacency matrix as its fundamental spectral data. Gutman defined the energy of Γ_G to be the total obtained by adding the absolute values of all eigenvalues of the graph. It is applied to the total π-electron energy approximation of molecules in the field of Chemistry. The molecules are viewed as graphs, with the vertices being carbon atoms and the hydrogen bonds between carbon atoms as edges. After years of dynamic research, the concept has been extended to several other types of graph matrices, such as the Randic matrix and harmonic matrix. The nature of these matrices is vertex-degree-based, whose entries obey the adjacency property between two vertices. Gutman et al. (2014) introduced the Randic energy of a graph and discussed the properties of the Randic energy of simple finite graphs.

Furthermore, the Quantitative Structure-Property Relationships (QSPR) study of vertex Zagreb adjacency energy, forgotten adjacency energy, and harmonic energy can be found in Hosamani et al. (2017). They introduced the definition of the harmonic matrix along with the vertex Zagreb adjacency and forgotten adjacency matrices. The bounds of harmonic energy were discussed in Jahanbani and Raz (2019). These results subsequently triggered Al-Roainee and Mahde (2024) to modify the harmonic matrix. Later, Gao et al. (2021) investigated the Randic energy of trees. One year later Cruz et al. (2022) derived the Randic energy formula of directed graph. In addition, Arizmendi and Huerta (2025) studied the relationship between graph energy and Randic index. The concept of graph energy has proven useful in estimating key thermodynamic properties of alkanes, including their boiling points, vaporization enthalpies, and critical temperatures, as well as pharmacological properties (Raza and Munir, 2024).

Additionally, a number of studies have been done on finding the bounds of the Randic energy. Milovanovic et al. (2014) discovered some graph energy bounds; however, three years later, Das and Elumalai (2017) corrected some errors in determining the lower bound. Later, Altındag (2018) obtained a criterion by which we can determine when two graphs are almost Randic equienergetic. Moreover, Allem et al. (2020) provided improved upper bounds of the Randic energy. For bipartite graphs, Sorgun et al. (2019) obtained the Randic energy and modified the graph such that it has the same Randic energy. Recently, Zhao et al. (2022) showed that the Randic energy for a tree is not greater than the Randic energy for a path on n vertices.

On the other hand, research related to the ordinary energy of the commuting graph has been done by several authors. You may find information on the Laplacian and the signless Laplacian spectrum of D_{2n} in Dutta and Nath (2018, 2021). Torktaz and Ashrafi (2019) and Wang et al. (2022) investigated commuting graphs of particular classes of finite groups and explicitly derived their normalized Laplacian and signless Laplacian spectra. Meanwhile, the energy of the commuting graph for D_{2n}, where $n \geq 3$ can be found in Romdhini et al. (2022) and Romdhini et al. (2025), which formulated the energy associated with the first Zagreb, GCD-degree, as well as degree exponent sum. In general, graphs constructed from the vertex set of a finite group, such as the prime graph (Keller et al., 2025), the prime coprime graph (Abdurahim et al., 2026; Romdhini and Nawawi, 2025), the equal-square graph (Rana et al., 2024), and the co-prime graph (Sehgal et al., 2021) have been extensively studied, particularly in relation to their spectral properties and topological indices.

However, a critical gap remains in the literature. Although degree-based energy concepts, such as Randic energy and harmonic energy, and commuting graphs of finite groups have been investigated independently, no direct connection has yet been established between commuting graphs and Randic or harmonic matrices. This limitation is particularly significant for dihedral groups, whose commuting graphs are isomorphic to certain graph classes with well-characterized vertex-degree distributions, precisely the structural setting in which Randic and harmonic matrices provide the most informative spectral descriptors. Although there are extensive results on the spectral and energy properties of commuting graphs, such as those based on the Zagreb matrix, the GCD-degree, the degree exponent sum, these approaches have not captured the Randic and harmonic matrices which are crucial for understanding the subtler structural variations in commuting graphs.

This study aims to analyze the relationship between graph energies resulting from those matrices that represent the commuting graph for dihedral groups. These results extend the theory of graph energy into a new algebraic setting and open avenues for further applications of degree-based spectral invariants in group-associated graphs.

2. EXPERIMENTAL SECTION

We first introduce the essential definitions and notation employed throughout this study. For clarity, the symbols and their corresponding meanings are summarized in the following table.

Table 1. Notation and its Definition

Symbol	Definition
D_{2n}	dihedral graph
$Z(D_{2n})$	center of D_{2n}
$C_{D_{2n}}(v)$	centralizer of $v \in D_{2n}$
Γ_{G_i}	commuting graph of $G_i \subseteq D_{2n}$
$R(\Gamma_{G_i})$	Randic matrix of Γ_{G_i}
$H(\Gamma_{G_i})$	harmonic matrix of Γ_{G_i}
$P_{R(\Gamma_{G_i})}(\lambda)$	characteristic polynomial of $R(\Gamma_{G_i})$
$P_{H(\Gamma_{G_i})}(\lambda)$	characteristic polynomial of $H(\Gamma_{G_i})$
λ	eigenvalue of the matrix
R_i	$i-$th row operation
C_i	$i-$th column operation
$Spec(\Gamma_{G_i})$	Spectrum of Γ_{G_i}
$E_R(\Gamma_{G_i})$	Randic energy of Γ_{G_i}
$E_H(\Gamma_{G_i})$	harmonic energy of Γ_{G_i}
d_v	degree of vertex v
K_n	complete graph on n vertices
$\bar{K}_n$	complement of K_n

A group is a mathematical structure endowed with a binary operation satisfying closed and associative conditions, containing an identity, and in which every element has a unique inverse (Aschbacher, 2012). For integers $n \geq 3$, the dihedral group of order $2n$, which models the rotational and reflective symmetries of a regular $n-$sided polygon, is defined by the presentation $D_{2n} = \left\langle a, b : a^n = b^2 = e, bab = a^{-1} \right\rangle$. The center of this group is trivial when n is odd, namely $Z\left(D_{2n}\right) = \{e\}$, whereas for even n it contains two elements and is given by $Z\left(D_{2n}\right) = \left\{e, a^{\frac{n}{2}}\right\}$. The centralizer of a rotation a^i in D_{2n} is is the cyclic subgroup generated by a, $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$, while the centralizer of a reflection $a^i b$ depends on the parity of n, $C_{D_{2n}}(a^i b) = \{e, a^i b\}$, if n is odd or $C_{D_{2n}}(a^i b) = \left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}$, if n is even.

For any subset G_i of D_{2n}, we define a graph Γ_{G_i}, having $G_i \backslash Z(G)$ as its vertices, and with an edge joining two distinct vertices of $G_i \backslash Z(G)$ that commute under the composition of D_{2n}. We refer to Γ_{G_i} as the commuting graph of G_i. This graph is a simple finite graph, without directed or multiple edges, and no loops, and was introduced by Brauer and Fowler (1955).

We now define the Randic and harmonic matrices n terms vertex degree information, where the degree d_{v_p} of v_p represents the total number of vertices to which v_p is adjacent.

Definition 2.1. (Gutman et al., 2014) Randic matrix (R) of Γ_{G_i} is given by $R(\Gamma_{G_i}) = [r_{pq}]_{n \times n}$ with each entry defined by

$$r_{pq} = \begin{cases} \frac{1}{\sqrt{d_{v_p} \cdot d_{v_q}}}, & \text{for adjacent } v_p \text{ and } v_q \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. (Hosamani et al., 2017) Harmonic matrix (H) of Γ_{G_i} is denoted by $H(\Gamma_{G_i})$
$= [h_{pq}]_{n \times n}$ with each entry defined by

$$h_{pq} = \begin{cases} \frac{2}{d_{v_p} + d_{v_q}}, & \text{for adjacent } v_p \text{ and } v_q \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the eigenvalues of the Randic matrix of Γ_{G_i}, $R = R(\Gamma_{G_i})$ is denoted by $\lambda_1, \lambda_2, \ldots, \lambda_m$ and written in decreasing order. The set $Spec(\Gamma_{G_i}) = \left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m}\right\}$ is called the $R-$spectrum of the commuting graph Γ_{G_i}, where $k_1, k_2, \ldots, k_m$ are the multiplicities of the respective eigenvalues. In accordance with the standard approach for extending the notion of graph energy to alternative matrix representations, the energy corresponding to the Randic matrix is defined as the sum of the absolute values of its eigenvalues and is referred to as the Randic energy of Γ_{G_i}, $E_R(\Gamma_{G_i})$. Furthermore, the above notations and terminologies can be applied analogously to harmonic energy, $E_H(\Gamma_{G_i})$.

In order to assist us in constructing the matrices and obtaining their characteristic polynomial of Γ_{G_i} as well as inspecting the graph with regard to its isomorphism, we need the following results.

Lemma 2.4.(Ramane and Shinde, 2017) Let the determinant

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix},$$

where a, b, c and d are real-valued parameters, I_n represents the identity matrix of dimension $n \times n$, and $J_{m \times n}$ is the all-ones matrix. Then, the determinant simplifies to the compact factorized form

$$(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}$$
$$\Big((\lambda - (n_1 - 1)\, a)\,(\lambda - (n_2 - 1)\, b) - n_1 n_2 cd\Big).$$

Theorem 2.5.(Gantmacher, 1959) If a square matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is divided into four block submatrices A, B, C, D, with $|A| \neq 0$, then the determinant of M can be calculated as

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A| \left|D - CA^{-1}B\right|.$$

A graph on n vertices where each vertex is connected to all others is referred to as the complete graph and is denoted by K_n. The complementary graph is denoted by $\bar{K}_n$. The lemma below states the eigenvalue spectrum of K_n, which will later be employed in the computation of the Randic and harmonic energies of Γ_{G_i}.

Lemma 2.5.(Brouwer and Haemers, 2012) The adjacency matrix of K_n is $(J - I)_n$, and its spectrum is $\{(n-1)^1, (-1)^{n-1}\}$.

We next examine certain subsets of D_{2n}. Let $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z\,(D_{2n})$, denote the collection of rotational elements outside the center, and let $G_2 = \{a^i b : 1 \leq i \leq n\}$, represents the set of reflections. Define $G = G_1 \cup G_2$. The following result describes the vertex degrees of Γ_G.

Theorem 2.6.(Romdhini et al., 2022) In Γ_G for $G = G_1 \cup G_2$,

1. the degree of a^i is $d_{a^i} = \begin{cases} n-2, & \text{if } n \text{ is odd} \\ n-3, & \text{if } n \text{ is even,} \end{cases}$ and
2. the degree of $a^i b$ is $d_{a^i b} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even.} \end{cases}$

Now, on the other hand, we also remark here that Γ_G, either $G = G_1$ or $G = G_2$, is isomorphic to some notable graph, which has been proved in Romdhini et al. (2022).

Theorem 2.7. (Romdhini et al., 2022) In Γ_G,

1. ff $G = G_1$, then $\Gamma_G \cong K_m$, where $m = |G_1|$, and
2. if $G = G_2$, then $\Gamma_G \cong \begin{cases} \bar{K}_n, & \text{if } n \text{ is odd} \\ 1 - \text{regular graph}, & \text{if } n \text{ is even.} \end{cases}$

3. RESULTS AND DISCUSSION

This section derives the formula of energy of the commuting graph Γ_G of dihedral groups, computed with respect to the associated Randic and harmonic matrices. We divide the nonabelian dihedral group, D_{2n}, into two cases, odd n and even n, since both have different centers and centralizers for reflection elements $a^i b$.

Theorem 3.1. In Γ_G,

1. if $G = G_1$, then the Randic energy of Γ_G, denoted as E_R and the harmonic energy of Γ_G, denoted as E_H, are $E_R(\Gamma_G) = E_H(\Gamma_G) = 2$, and
2. if $G = G_2$, then the Randic and harmonic energies of Γ_G are

$$E_R(\Gamma_G) = E_H(\Gamma_G) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

Proof.

1. For odd values of n, Theorem 2.7 (1) establishes that, upon excluding the identity element from the center $Z(D_{2n})$, Γ_G with $G = G_1$ is isomorphic to K_m, where $m = |G_1| = n - 1$. Consequently, by Theorem 2.6 (1), every vertex in Γ_G has degree $(n-2)$. By Definition 2.1 and 2.2, an $(n-1) \times (n-1)$ Randic matrix of Γ_G can be formed, $R(\Gamma_G) = \left[r_{pq}\right]$, and harmonic matrix of Γ_G, $H(\Gamma_G) = \left[h_{pq}\right]$ in which entries are $r_{pq} = \frac{1}{\sqrt{(n-2)(n-2)}} = \frac{1}{n-2}$ and $h_{pq} = \frac{2}{(n-2)+(n-2)} = \frac{1}{n-2}$, whenever v_p and v_q are adjacent, and $r_{pq} = h_{pq} = 0$ otherwise. As a result, the two matrices share an identical structural representation, given by the following form:

$$R(\Gamma_G) = H(\Gamma_G) =$$

$$\begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{array} \begin{array}{c} \begin{array}{cccc} a & a^2 & \ldots & a^{n-1} \end{array} \\ \begin{pmatrix} 0 & \frac{1}{n-2} & \ldots & \frac{1}{n-2} \\ \frac{1}{n-2} & 0 & \ldots & \frac{1}{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-2} & \frac{1}{n-2} & \ldots & 0 \end{pmatrix} \end{array}$$

$$= \frac{1}{n-2} \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}.$$

Clearly, $R(\Gamma_G)$ and $H(\Gamma_G)$ can be expressed as scalar multiples of the adjacency matrix of K_{n-1}, with scaling factor $\frac{1}{n-2}$. Also note that from Lemma 2.5, both matrices are the product of $\frac{1}{n-2}$ and $(J-I)_{n-1}$. Furthermore, $Spec(K_{n-1}) = \left\{(n-2)^1, (-1)^{n-2}\right\}$ which results in the adjacency energy of K_{n-1} equals $|n-2| + (n-2)|-1| = 2(n-2)$. Consequently, both the Randic and harmonic energies of Γ_G are equal to $\frac{1}{n-2} \cdot 2(n-2) = 2$.

When n is an even integer, Theorem 2.7 (1) implies that, after excluding the elements e and $a^{\frac{n}{2}}$ from $Z(D_{2n})$, Γ_G with $G = G_1$ is isomorphic to K_m where $m = |G_1| = n-2$. As a consequence, Theorem 2.6 (1) ensures that each vertex in this graph has degree $(n-3)$. According to Definition 2.2 and 2.1, Randic matrix $R(\Gamma_G) = \left[r_{pq}\right]$ is an $(n-2)\times(n-2)$ matrix matrix whose nonzero entries correspond to adjacent vertices and are equal to $r_{pq} = \frac{1}{\sqrt{(n-3)(n-3)}} = \frac{1}{n-3}$, and $r_{pq} = 0$ otherwise. The harmonic matrix $H(\Gamma_G) = \left[h_{pq}\right]$ has the same dimension and structure, since its nonzero entries also take the value $h_{pq} = \frac{2}{(n-3)+(n-3)} = \frac{1}{n-3}$, whenever two vertices are adjacent, and $h_{pq} = 0$ otherwise. We then obtain as follows

$$R(\Gamma_G) = H(\Gamma_G) =$$

$$\begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{array}
\begin{array}{c}
\begin{array}{cccc} a & a^2 & \dots & a^{n-1} \end{array} \\
\begin{pmatrix}
0 & \frac{1}{n-3} & \dots & \frac{1}{n-3} \\
\frac{1}{n-3} & 0 & \dots & \frac{1}{n-3} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n-3} & \frac{1}{n-3} & \dots & 0
\end{pmatrix}
\end{array}$$

$$= \frac{1}{n-3}\begin{pmatrix}
0 & 1 & \dots & 1 \\
1 & 0 & \dots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \dots & 0
\end{pmatrix}.$$

It is apparent that, $R(\Gamma_G)$ and $H(\Gamma_G)$ obtained by multiplying the adjacency matrix of K_{n-2} (or $J_{n-2} - I_{n-2}$) by the scalar $\frac{1}{n-3}$. By Lemma 2.5, $Spec(K_{n-2})$ is given by $\left\{(n-3)^1, (-1)^{n-3}\right\}$. Hence, the adjacency energy of K_{n-2} is $|n-3| + (n-3)|-1| = 2(n-3)$, consequently the Randic and harmonic energies of Γ_G will be $\frac{1}{n-3} \cdot 2(n-3) = 2$.

2. For odd values of n, Theorem 2.7 (2) shows that, when $G = G_2$, $\Gamma_G \cong \bar{K}_n$. Thus Γ_G is a disconnected graph with n isolated vertices, which means that there are no adjacent vertices in Γ_G. By Definition 2.1 and 2.2, the entries of $R(\Gamma_G)$ and $H(\Gamma_G)$ are all zero. Consequently, each matrix reduces to the $n \times n$ zero matrix, which implies that $E_R(\Gamma_G) = E_H(\Gamma_G) = 0$.

   For even values of n. Theorem 2.7 (2), asserts that, when $G = G_2$, Γ_G is a 1-regular. In accordance with Definition 2.1 and 2.2, one can therefore form the associated $n \times n$ Randic and harmonic matrices, denoted by $R(\Gamma_G) = \left[r_{pq}\right]$ and $H(\Gamma_G) = \left[h_{pq}\right]$, respectively. For any pair of adjacent vertices v_p and v_q the corresponding matrix entries satisfy $r_{pq} = \frac{1}{\sqrt{1\cdot 1}} = 1$, and $h_{pq} = \frac{2}{1+1} = 1$, while all remaining entries are equal to zero.

$$R(\Gamma_G) = H(\Gamma_G) =$$

$$\begin{array}{c} \\ b \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ \vdots \\ a^{n-1}b \end{array}
\begin{array}{c}
\begin{array}{cccccc} b & \dots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \dots & a^{n-1}b \end{array} \\
\begin{pmatrix}
0 & \dots & 0 & 1 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 1 \\
1 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 1 & 0 & \dots & 0
\end{pmatrix}
\end{array}$$

$$= \begin{pmatrix} 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{pmatrix}.$$

The two matrices are identical, so they have the same characteristic polynomial.

$$P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = \begin{vmatrix} \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix}. \tag{1}$$

Consider R_i as the $i-$th row and R_i' represent the row obtained after performing a row operation on either $P_{R(\Gamma_G)}(\lambda)$ or $P_{H(\Gamma_G)}(\lambda)$. By replacing each row $R_{\frac{n}{2}+i}$ with with the difference between the $\frac{n}{2}+i$-th row and the i-th row, that is, $R'_{\frac{n}{2}+i} = R_{\frac{n}{2}+i} - R_i$, for $1 \le i \le \frac{n}{2}$, consequently, we derive

$$P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = \begin{vmatrix} \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -(\lambda+1) I_{\frac{n}{2}} & (\lambda+1) I_{\frac{n}{2}} \end{vmatrix}. \tag{2}$$

Let C_i denote the $i-$th column and C_i' is the updated $i-$th column obtained after applying a column operation to $P_{R(\Gamma_G)}(\lambda)$ or $P_{H(\Gamma_G)}(\lambda)$. Replacing each column according to C_i with the sum of the i-th and $\frac{n}{2}+i$-th columns, that is, $C_i' = C_i + C_{\frac{n}{2}+i}$, for every $1 \le i \le \frac{n}{2}$, then Equation (2) can be simplified as follows:

$$P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda-1) I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda+1) I_{\frac{n}{2}} \end{vmatrix}. \tag{3}$$

Then, the right-hand side of Equation (3) is an upper triangular matrix. The determinant is obtained by multiplying all diagonal entries of $P_{R(\Gamma_G)}(\lambda)$. By direct calculation, we now obtain

$$P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = \left|(\lambda-1) I_{\frac{n}{2}}\right| \left|(\lambda+1) I_{\frac{n}{2}}\right| = (\lambda+1)^{\frac{n}{2}} (\lambda-1)^{\frac{n}{2}}.$$

Consequently,

$$E_R(\Gamma_G) = E_H(\Gamma_G) = \binom{n}{2}|1| + \binom{n}{2}|-1| = n.$$

Theorem 3.2. In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, the characteristic polynomials of $R(\Gamma_G)$ and $H(\Gamma_G)$ are

1. $P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = \left(\lambda + \frac{1}{n-2}\right)^{n-2} \lambda^n(\lambda - 1)$, for odd n, and
2. $P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = \left(\lambda + \frac{1}{n-3}\right)^{n-3} (\lambda + 1)^{\frac{n}{2}}(\lambda - 1)^{\frac{n}{2}+1}$, for even n.

Proof

1. Suppose n is odd. By Theorem 2.6, it follows that each element a^i for $1 \le i < n$ has degree $n-2$, whereas every element of a^ib, with $1 \le i \le n$, has degree zero. Given that $Z(D_{2n}) = \{e\}$, Γ_G where $G = G_1 \cup G_2$, consists of $2n-1$ vertices. Among these, $n-1$ vertices correspond to the elements a^i for $1 \le i < n$, and the remaining n vertices correspond to the elements a^ib for $1 \le i \le n$. Based on Definition 2.1 and 2.2, the Randic matrix for Γ_G, denoted by $R(\Gamma_G)$ is a square matrix of order $2n-1$ matrix in which entries r_{pq} are given as follows:
   (a) $r_{pq} = \frac{1}{\sqrt{(n-2)\cdot(n-2)}} = \frac{1}{n-2}$, for all distinct p and q with $1 \le p, q \le n-1$,
   (b) $r_{pq} = 0$, otherwise,

   whereas harmonic matrix for Γ_G, $H(\Gamma_G)$, is a matrix of the same dimension as $R(\Gamma_G)$, in which entries h_{pq} are obtained as below:
   (a) $h_{pq} = \frac{2}{(n-2)+(n-2)} = \frac{1}{n-2}$, for all distinct p and q satisfying $1 \le p, q \le n-1$,
   (b) $h_{pq} = 0$, otherwise.

   As a straightforward consequence, $R(\Gamma_G) = H(\Gamma_G)$ and we may write them as

$$\begin{array}{c|cccccccc}
 & a & a^2 & \dots & a^{n-2} & a^{n-1} & b & \dots & a^{n-1}b \\
\hline
a & 0 & \frac{1}{n-2} & \dots & \frac{1}{n-2} & \frac{1}{n-2} & 0 & \dots & 0 \\
a^2 & \frac{1}{n-2} & 0 & \dots & \frac{1}{n-2} & \frac{1}{n-2} & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a^{n-2} & \frac{1}{n-2} & \frac{1}{n-2} & \dots & 0 & \frac{1}{n-2} & 0 & \dots & 0 \\
a^{n-1} & \frac{1}{n-2} & \frac{1}{n-2} & \dots & \frac{1}{n-2} & 0 & 0 & \dots & 0 \\
b & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a^{n-1}b & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0
\end{array}$$

$$= \begin{pmatrix} \frac{1}{n-2}(J-I)_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_n \end{pmatrix}.$$

   Accordingly, the characteristic formula of $R(\Gamma_G)$ and $H(\Gamma_G)$ are

$$P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = \begin{vmatrix} \left(\lambda + \frac{1}{n-2}\right) I_{n-1} - \frac{1}{n-2} J_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

   Applying Lemma 2.3, with the parameter choices $a = \frac{1}{n-2}$, $b = c = d = 0$, $n_1 = n-1$ and $n_2 = n$, we obtain a simplified form of the characteristic polynomial as follows:

$$\left(\lambda + \frac{1}{n-2}\right)^{n-2} \lambda^n(\lambda - 1).$$

2. Let us now examine the case in which n is an even integer. It is apparent from Theorem 2.6 that $d_{a^i} = n-3$ for $1 \le i < n$, while every vertex of the form a^ib has degree 1, for $1 \le i \le n$. Moreover, since the center of D_{2n} is given by $\{e, a^{\frac{n}{2}}\}$, Γ_G, where $G = G_1 \cup G_2$, contains $2n-2$ vertices. Of these, $n-2$ correspond to the elements a^i with $1 \le i < n$ and $i \ne \frac{n}{2}$, while the remaining n vertices arise from the elements a^ib for $1 \le i \le n$. Therefore, by Definition 2.1 and 2.2, both Randic $R(\Gamma_G) = [r_{pq}]$ and harmonic $H(\Gamma_G) = [h_{pq}]$ matrices are determined as follows::
   (a) $r_{pq} = \frac{1}{\sqrt{(n-3)\cdot(n-3)}} = \frac{1}{n-3}$, and $h_{pq} = \frac{2}{(n-3)+(n-3)} = \frac{1}{n-3}$ for $p \ne q$, and $1 \le p, q \le n-2$,
   (b) $r_{pq} = h_{pq} = 1$, for $n-1 \le p \le n-2+\frac{n}{2}$, and $q = \frac{n}{2}+p$,
   (c) $r_{pq} = h_{pq} = 1$, for $n-1 \le q \le n-2+\frac{n}{2}$, and $p = \frac{n}{2}+q$,
   (d) $r_{pq} = 0$, otherwise,

   As a result, $R(\Gamma_G) = H(\Gamma_G)$ and be as follows:
   Accordingly, the characteristic polynomials of $R(\Gamma_G)$ and $H(\Gamma_G)$ can be formulated through the determinant representation

$$\begin{aligned} P_{R(\Gamma_G)}(\lambda) &= P_{H(\Gamma_G)}(\lambda) \\ &= \begin{vmatrix} \left(\lambda + \frac{1}{n-3}\right) I_{n-2} - \frac{1}{n-3} J_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix} \\ &= \begin{vmatrix} A_{n-2} & B_{(n-2)\times n} \\ C_{n\times(n-2)} & D_n \end{vmatrix}. \end{aligned}$$

   Having B and C as blocks with zero entries, we use Theorem 2.4 to give

$$\begin{aligned} P_{R(\Gamma_G)}(\lambda) &= P_{H(\Gamma_G)}(\lambda) \\ &= |A||D - CA^{-1}B| = |A||D|. \end{aligned} \tag{4}$$

   We first examine the determinant $|A|$ by using Lemma 2.3, with $a = b = c = d = \frac{1}{n-3}$, and $n_1 = n_2 = \frac{n-2}{2}$, then

$$|A| = \left(\lambda + \frac{1}{n-3}\right)^{n-3} (\lambda - 1). \tag{5}$$

   Whereas, for determinant $|D|$, in an analogous manner as in the proof of Equation (1), we know

$$|D| = (\lambda + 1)^{\frac{n}{2}}(\lambda - 1)^{\frac{n}{2}}. \tag{6}$$

   By substituting Equation (5) and (6) into (4), the charac-

$$\begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-2} \\ a^{n-1} \\ b \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ \vdots \\ a^{n-1}b \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} a & a^2 & \dots & a^{n-2} & a^{n-1} & b & \dots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \dots & a^{n-1}b \end{array} \\
\begin{pmatrix}
0 & \frac{1}{n-3} & \dots & \frac{1}{n-3} & \frac{1}{n-3} & 0 & \dots & 0 & 0 & \dots & 0 \\
\frac{1}{n-3} & 0 & \dots & \frac{1}{n-3} & \frac{1}{n-3} & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n-3} & \frac{1}{n-3} & \dots & 0 & \frac{1}{n-3} & 0 & \dots & 0 & 0 & \dots & 0 \\
\frac{1}{n-3} & \frac{1}{n-3} & \dots & \frac{1}{n-3} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 \\
0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0
\end{pmatrix}
\end{array}$$

$$= \begin{pmatrix} \frac{1}{n-3}(J-I)_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{pmatrix}.$$

teristic formula of $R(\Gamma_G)$ and $H(\Gamma_G)$ is as follows:

$$P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda)$$
$$= \left(\lambda + \frac{1}{n-3}\right)^{n-3} (\lambda+1)^{\frac{n}{2}} (\lambda-1)^{\frac{n}{2}+1}.$$

Considering the result in the preceding theorem, we are now able to have results on the spectrum as well as the Randic and harmonic energies of Γ_G.

Theorem 3.3 In Γ_G for $G = G_1 \cup G_2$, the Randic and harmonic energies of Γ_G are

$$E_R(\Gamma_G) = E_H(\Gamma_G) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 2+n, & \text{if } n \text{ is even.} \end{cases}$$

Proof According to Theorem 3.2 (1) in the case where n is odd, both $P_{R(\Gamma_G)}(\lambda)$ and $P_{H(\Gamma_G)}(\lambda)$ have exactly three distinct eigenvalues. Specifically, the spectrum consists of $\lambda_1 = 1$ occurring with multiplicity 1, the eigenvalue $\lambda_2 = 0$ with multiplicity n, and a third $\lambda_3 = -\frac{1}{n-2}$ appearing with multiplicity $n-2$. Thus, we get the spectrum of Γ_G is

$$Spec(\Gamma_G) = \left\{(1)^1, (0)^n, \left(-\frac{1}{n-2}\right)^{n-2}\right\}.$$

By direct calculation, we now obtain Randic and harmonic energies

$$E_R(\Gamma_G) = E_H(\Gamma_G) = |1| + (n)\,|0| + (n-2)\left|-\frac{1}{n-2}\right| = 2.$$

In the case where n is even, Theorem 3.2 (2) shows that $P_{R(\Gamma_G)}(\lambda)$ and $P_{H(\Gamma_G)}(\lambda)$ admit exactly three distinct eigenvalues. Specifically, $\lambda_1 = 1$ of multiplicity $\frac{n}{2}+1$, $\lambda_2 = -\frac{1}{n-3}$ with multiplicity $n-3$, and $\lambda_3 = -1$ of multiplicity $\frac{n}{2}$. Then the spectrum of Γ_G is

$$Spec(\Gamma_G)) = \left\{(1)^{\frac{n}{2}+1}, \left(-\frac{1}{n-3}\right)^{n-3}, (-1)^{\frac{n}{2}}\right\}.$$

By observing the spectrum, it is clear that the Randic and harmonic energies of Γ_G is

$$E_R(\Gamma_G) = E_H(\Gamma_G) = \left(\frac{n}{2}+1\right)|1|$$
$$+ (n-3)\left|-\frac{1}{n-3}\right| + \left(\frac{n}{2}\right)|-1| = 2+n.$$

At first glance, the fact that the Randic energy and the harmonic energy of the commuting graph Γ_G are equal may seem unexpected, since these two energies are derived from different degree-based matrices. However, this equality turns out to be a natural consequence of the way the commuting graph of a dihedral group is structured.

The immediate result from Theorem 3.1 and 3.3 can be stated as follows.

Corollary 3.4. In Γ_G for $G = G_1 \cup G_2$,

1. $E_R(\Gamma_G) = E_R(\Gamma_{G_1})$ and $E_H(\Gamma_G) = E_H(\Gamma_{G_1})$, for odd n, and
2. $E_R(\Gamma_G) = E_R(\Gamma_{G_1}) + E_R(\Gamma_{G_2})$ and $E_H(\Gamma_G) = E_H(\Gamma_{G_1}) + E_H(\Gamma_{G_2})$, for even n.

When n is odd, the commuting graph of G has the same energy as that of Γ_{G_1} for both Randic and Harmonic matrices. This follows from the fact that Γ_G contains a single connected component isomorphic to a complete graph on $n-1$ vertices, while all remaining vertices are isolated. In contrast, when n is even, the commuting graph of Γ_G, decomposes into two disconnected components, and its energy is therefore given by the sum of the energies Γ_{G_1} and Γ_{G_2}.

The following is an example of the Randic and harmonic energies of the commuting graph for D_{2n} for $n = 4$ and $n = 5$.

Example 3.5 Let $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Define the subset $G_1 = \{a, a^3\}$ and $G_2 = \{b, ab, a^2b, a^3b\}$. The centralizers of a in D_8 coincides with that of a^3 and is given by $\{e, a, a^2, a^3\}$. Similarly, the elements b and a^2b share the same centralizer $\{e, a^2, b, a^2b\}$, while the centralizers of ab and a^3b are identical and equal to $\{e, a^2, ab, a^3b\}$. Using these centralizer structures for the elements of $G = G_1 \cup G_2$, the corresponding commuting graph Γ_G, is illustrated in Figure 1.

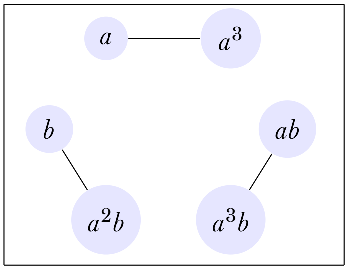

Figure 1. Commuting Graph Γ_G Associated with Subsets of D_8

Moreover, Γ_G can be represented as Randic, $R(\Gamma_G)$ and harmonic, $H(\Gamma_G)$ matrices as follows:

$$R(\Gamma_G) = H(\Gamma_G) = \left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$
$$= \begin{pmatrix} (J-I)_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & I_2 & 0_2 \end{pmatrix}.$$

Here, with the aid of MAPLE, the characteristic formula of both $R(\Gamma_G)$ and $H(\Gamma_G)$ is $P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = (\lambda + 1)^3(\lambda - 1)^3$, and consequently, $Spec(\Gamma_G) = \{(1)^3, (-1)^3\}$. Therefore, the Randic and harmonic energies of Γ_G are given as

$$E_R(\Gamma_G) = E_H(\Gamma_G) = (3)|1| + (3)|-1| = 6 = 2 + 4.$$

Example 3.6. Consider the dihedral group D_{10}, whose elements are $\{e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$. Its center is trivial, that is, $Z(D_{10}) = \{e\}$. Let G_1 denote the subset consisting of the rotational elements $\{a, a^2, a^3, a^4\}$, and let G_2 be the subset $\{b, ab, a^2b, a^3b, a^4b\}$. The centralizer of each reflection a^ib in D_{10} is $\{e, a^ib\}$, while the centralizer of any rotation a^i, is the cyclic subgroup generated by a. Based on these observations, the commuting graph associated with $G = G_1 \cup G_2$ is depicted in Figure 2.

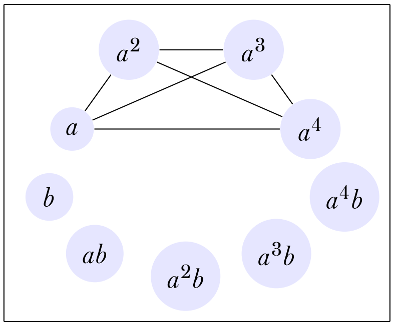

Figure 2. Commuting Graph Γ_G Associated with Subsets of D_{10}

Moreover, Γ_G can be represented as Randic, $R(\Gamma_G)$ and harmonic, $H(\Gamma_G)$ matrices as follows:

$$R(\Gamma_G) = H(\Gamma_G) = \left(\begin{array}{cccc|ccccc} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$= \begin{pmatrix} \frac{1}{3}(J-I)_4 & 0_{4\times 5} \\ 0_{5\times 4} & 0_5 \end{pmatrix}.$$

Hence, with the assistance of MAPLE, we derive the characteristic polynomial of $R(\Gamma_G)$ and $H(\Gamma_G)$ as follows:

$$P_{R(\Gamma_G)}(\lambda) = P_{H(\Gamma_G)}(\lambda) = (\lambda + \frac{1}{3})^3 \lambda^5 (\lambda - 1),$$

and it is relatively straightforward to have $Spec(\Gamma_G) = \{(1)^1, (0)^5, (-\frac{1}{3})^3\}$. By direct computation, we then obtain the Randic and harmonic energies of Γ_G, $E_R(\Gamma_G) = E_H(\Gamma_G) = (1)|1| + (5)|0| + (3)|-\frac{1}{3}| = 2$.

4. CONCLUSIONS

The Randic and harmonic energies of the commuting graph, Γ_G, are obtained for G being the subsets of non-abelian dihedral groups, D_{2n}, either $G = G_1$, $G = G_2$, or $G = G_1 \cup G_2$, the union of both sets. By observing the properties of the Randic and harmonic matrices of Γ_G, we observe that the Randic energy is identically equal to the harmonic energy, where it is either 2, for odd n, or $2+n$, for even n. Future research may explore whether this phenomenon persists for commuting graphs of other finite groups, such as generalized dihedral groups, nilpotent groups, or solvable groups. Additionally, extending the analysis to modified Randić indices, weighted commuting graphs, or other degree-based matrices may reveal finer distinctions between algebraic graph structures. These directions open promising avenues for further integration of algebraic graph theory and spectral energy concepts.

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