Science and Technology Indonesia
e-ISSN:2580-4391 p-ISSN:2580-4405
Vol. 10, No. 1, January 2025

Research Paper

Improved Maximum Likelihood Estimator of the Extended Rama Distribution with Application to Lifetime Data

Chanakan Sungboonchoo[1*]

[1]Department of Statistics, Faculty of Science, Silpakorn University, Nakhon Pathom, 73000, Thailand
*Corresponding author: sungboonchoo_c@silpakorn.edu

Abstract
Lifetime data are involved in numerous applied sciences, and the extended Rama (ER) distribution can be used to model such data. The maximum likelihood method is widely used for estimating the parameters of any distribution, particularly with large sample sizes. However, its effectiveness diminishes for small or moderate sample sizes due to the potential for biased estimates. This study improves the maximum likelihood estimator (MLE) of the extended Rama distribution by using two bias-corrected methods based on the Cox-Snell and parametric bootstrap approaches. Monte Carlo simulation was examined in terms of average bias and root mean square error (RMSE). The results indicate that the proposed bias-corrected estimators perform well in reducing both bias and root mean square error, thereby improving the accuracy of the estimates. Conversely, the maximum likelihood estimator exhibits relatively poor performance. Overall, the parametric bootstrap method outperformed the others, even when applied to small and moderate sample sizes. Additionally, the bias-corrected estimators were applied to a real dataset.

Keywords
Maximum Likelihood Estimator, Extended Rama Distribution, Cox-Snell Bias-Correction, Bootstrap Bias-Correction, Monte Carlo Simulation

Received: 6 May 2024, Accepted: 11 October 2024
https://doi.org/10.26554/sti.2025.10.1.80-87

1. INTRODUCTION

The analysis and modeling of lifetime data are important in many applied sciences, such as engineering, insurance, and biological and medical sciences. Lifetime data represents a valuable form of information that involves observing the time it takes for a specific event to occur. The event may be the failure of a product, the development or remission of symptoms, or even the death of an individual. The time taken for these events to occur is known as lifetimes within the fields of study (Deshpande and Purohit, 2005; Lawless, 2002). The two well-known one-parameter lifetime distributions for modeling lifetime data are the exponential and Lindley distributions (Lindley, 1958). However, there are many scenarios where these two distributions are unsuitable for accurately fitting various lifetime data compared to other lifetime distributions (Shanker et al., 2015, 2017).

Many researchers developed the one-parameter lifetime distribution of the sum of random variables and their applications based on mixture distributions for fitting real data. Shanker (2017a) presented the Rama distribution as a two-component mixture of an exponential distribution with one parameter (θ) and a gamma ($4,\theta$) distribution. Rama distribution is one-parameter lifetime distribution and skewed to the right. The probability density function (PDF) of the Rama distribution is given by

$$f(x|\theta) = \frac{\theta^4}{\theta^3+6}(1+x^3)e^{-\theta x}, x > 0, \theta > 0,$$

and the corresponding cumulative distribution function (CDF) can be obtained as

$$F(x|\theta) = 1 - \left[1 + \frac{\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x}{\theta^3 + 6}\right] e^{-\theta x}, x > 0, \theta > 0.$$

Assessing the performance of the Rama distribution involves conducting goodness-of-fit tests (Turhan, 2020) on actual lifetime data and comparing it with various other lifetime distributions. The results indicate that the Rama distribution provides a more accurate characterization compared to the competing distributions analyzed in this study. Alhyasat et al. (2021) proposed a new lifetime distribution named the extended Rama (ER) distribution. This model was achieved by using the sum of two independent randomly distributed Rama variables. The probability density function (PDF) of the ER distribution in

Equation (1) is given by

$$f(x|\theta) = \frac{\theta^8}{(\theta^3+6)^2}\left(x + \frac{x^4}{2} + \frac{x^7}{140}\right)e^{-\theta x}, x > 0, \theta > 0. \quad (1)$$

The corresponding cumulative distribution function (CDF) can be obtained as

$$F(x|\theta) = 1 - \left(\frac{x^2\theta^2[42x^3\theta^3+7x^4\theta^4+x^5\theta^5+(3+\theta^3)(840+280x\theta+70x^2\theta^2)]}{140(6+\theta^3)^2} + 1 + x\theta\right)e^{(-\theta x)}, x > 0, \theta > 0.$$

Figure 1 illustrates the PDF of the ER distribution, which is skewed to the right over the interval $(0, 2)$ for specific parameter values of θ. The extent of this skewness varies with different parameter values. This research compares the performance of the maximum likelihood estimator for the parameter of the extended Rama distribution with different one-parameter lifetime distributions. The extended Rama distribution provides a better fit than Rama (Shanker, 2017a), exponential, Rani (Shanker, 2017b), and Maxwell length-biased (Saghir et al., 2017) distributions for modeling real lifetime data.

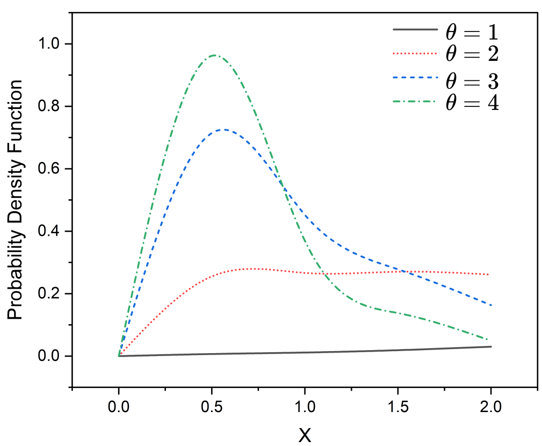

Figure 1. The PDF of the ER Distribution with $\theta = 1, 2, 3, 4$

In statistical inference, the most popular method for estimating the parameters of any probability distribution is the maximum likelihood method (Millar, 2011). This method holds intuitive appeal and possesses several essential properties that rely on a large sample size. For instance, it is asymptotically unbiased, consistent, and asymptotically normally distributed (Lehmann, 1999). Nevertheless, the maximum likelihood method results in biased estimates when utilized with small or moderate sample sizes. Therefore, this technique is unsuitable. The problem of estimating parameters for small or moderate sample sizes using the maximum likelihood method is interesting to consider in statistical inference.

Many researchers develop maximum likelihood estimators of the parameters of various distributions, especially in the context of small and moderate sample sizes. The distributions that have been studied to improve the maximum likelihood estimators are as follows: the Gompertz distribution, the Maxwell distribution, the Weibull distribution, the generalized half-normal distribution, the unit-gamma distribution, the inverse Weibull distribution, the unit-Lindley distribution, the unit-Weibull distribution, and the zero-inflated Poisson distribution (Al-Shomrani, 2023; Maghami and Bahrami, 2020; Makalic and Schmidt, 2023; Mazucheli et al., 2018; Mazucheli and Dey, 2018; Mazucheli et al., 2019, 2020, 2021; Schwartz and Giles, 2016).

To the best of our understanding, the literature has not examined the adjustment of the maximum likelihood estimator in cases of small and moderate sample sizes using bias-corrected estimators for the extended Rama distribution. It is important to note that the maximum likelihood estimator does not have a closed-form solution. Therefore, finding the MLE for the extended Rama distribution requires the use of numerical methods.

This research proposes two bias-correction methods aimed at reducing the bias of the maximum likelihood estimator, progressing from first-order bias correction ($\mathcal{O}(n^{-1})$) to second-order bias correction ($\mathcal{O}(n^{-2})$) for the parameter of the extended Rama distribution in cases of small and moderate sample sizes. First, this study focuses on the analytical technique introduced by Cox and Snell (1968), referred to as the "corrective" approach. This method corrects the bias of the maximum likelihood estimator to the second order by subtracting the bias from the true value of the maximum likelihood estimator. Second, the parametric bootstrap technique suggested by Efron (1982) is employed, which also achieves second-order bias correction. This technique numerically applies bias correction without necessitating the derivation of analytical expressions for the bias function.

A simulation study using the Monte Carlo method was conducted to compare the performance of these two bias correction methods with the classical maximum likelihood estimator. The results were used to determine the best performing method based on the average bias and the root mean square error. Furthermore, the maximum likelihood estimators of these methods can be applied to lifetime data.

2. EXPERIMENTAL SECTION

This section explains the MLE of the ER distribution. We define $x = (x_1, \ldots, x_n)$as a random sample of size n from the ER distribution, with the probability density function (pdf) provided in Equation (1). Then, the log-likelihood function of θ in (2) can be written as:

$$lL(\theta|x) = 8nlog\theta - 2nlog(6+\theta^3) - n\theta\bar{x} + \Sigma_{i=1}^{n} log\left(x_i + \frac{x_i^4}{2} + \frac{x_i^7}{140}\right). \quad (2)$$

Table 1. Estimated Average Bias (Root Mean Square Error) for $\theta = 1, 2$

θ	n	Estimator of θ $\hat{\theta}_{\text{MLE}}$	$\hat{\theta}_{\text{BCMLE}}$	$\hat{\theta}_{\text{BOOT}}$
1	10	0.9999425 (0.9999425)	0.00009698657 (0.009698657)	0.00009998981 (0.009998981)
	20	0.9999452 (0.9999452)	0.00009849187 (0.009849187)	0.00009999641 (0.009999641)
	30	0.9999472 (0.9999472)	0.00009899281 (0.009899281)	0.00009999605 (0.009999605)
	40	0.9999485 (0.9999485)	0.00009924315 (0.009924315)	0.00009999056 (0.009999056)
	50	0.9999496 (0.9999496)	0.00009939338 (0.009939338)	0.00009999769 (0.009999769)
2	10	-0.0000640486 (0.0000651052)	-0.000003012388 (0.0003012388)	-0.000000009407745 (0.0000009407745)
	20	-0.00006693072 (0.00006762422)	-0.000001512029 (0.0001512029)	-0.000000007015819 (0.0000007015819)
	30	-0.00006815157 (0.00006862966)	-0.000001009846 (0.0001009846)	-0.000000008937736 (0.0000008937736)
	40	-0.00006915947 (0.0000694665)	-0.0000007562481 (0.00007562481)	-0.000000007818198 (0.0000007818198)
	50	-0.0000696071 (0.00006983581)	-0.0000006092073 (0.00006092073)	-0.000000006770406 (0.0000006770406)

Table 2. Estimated Average Bias (Root Mean Square Error) for $\theta = 3, 4$

θ	n	Estimator of θ $\hat{\theta}_{\text{MLE}}$	$\hat{\theta}_{\text{BCMLE}}$	$\hat{\theta}_{\text{BOOT}}$
3	10	-1.000059 (1.000059)	-0.0001030128 (0.01030128)	-0.0001000092 (0.01000092)
	20	-1.000059 (1.000059)	-0.0001015121 (0.01015121)	-0.0001000039 (0.01000039)
	30	-1.000058 (1.000058)	-0.0001010072 (0.01010072)	-0.0001000028 (0.01000028)
	40	-1.000057 (1.000057)	-0.0001007566 (0.01007566)	-0.0001000043 (0.01000043)
	50	-1.000056 (1.000056)	-0.0001006093 (0.01006093)	-0.0001000091 (0.01000091)
4	10	-2.000052 (2.000052)	-0.0002030157 (0.02030157)	-0.0002000041 (0.02000041)
	20	-2.000053 (2.000053)	-0.0002015082 (0.02015082)	-0.0002000034 (0.02000034)
	30	-2.000053 (2.000053)	-0.0002010068 (0.02010068)	-0.0002000038 (0.02000038)
	40	-2.000052 (2.000052)	-0.0002007589 (0.02007589)	-0.000200004 (0.0200004)
	50	-2.000052 (2.000052)	-0.0002006057 (0.02006057)	-0.0002000029 (0.02000029)

Equation (3) shows the differentiation of Equation (2) with respect to θ in order to obtain the maximum likelihood estimate $\hat{\theta}_{MLE}$ of θ. The estimate $\hat{\theta}_{MLE}$ is obtained by solving the

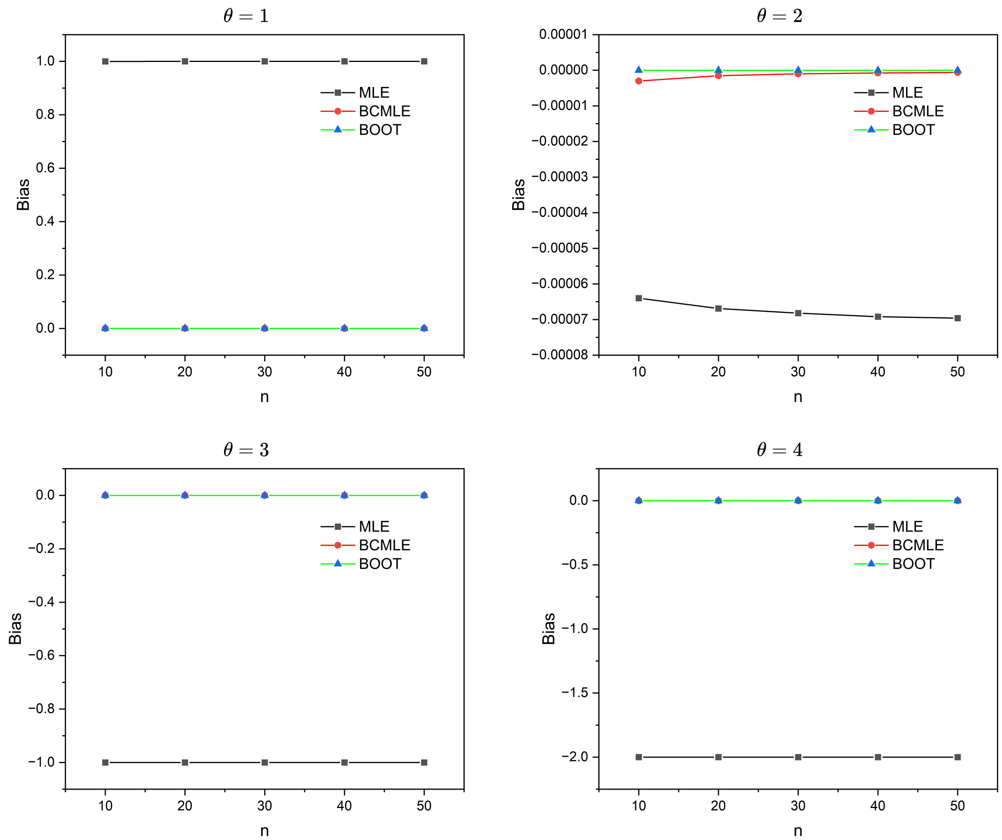

Figure 2. Plots Comparing the Average Bias of Three Different Estimation Methods for $\theta = 1, 2, 3, 4$

following Equation (3)

$$\frac{\partial}{\partial\theta} lL(\theta|x) = \frac{8n}{\theta} - \frac{6n\theta^2}{6+\theta^3} - n\bar{x} = 0. \quad (3)$$

Since there is no explicit solution for Equation (3), the numerical solution for the nonlinear equation can be obtained by using the optimize function in the R program.

2.1 Bias-Corrected MLEs

In finite samples, maximum likelihood estimators are typically biased of the order ($\mathcal{O}(n^{-1})$). For this reason, we use the following two bias-correction methods to reduce the bias of the MLEs of the ER distribution to order ($\mathcal{O}(n^{-2})$). We examine the technique proposed by Cox and Snell (1968) and the parametric bootstrap technique suggested by Efron (1982).

2.1.1 Cox-Snell Method

Define $lL(\theta|x)$ as the log-likelihood function derived from a sample of n observations, characterized by a p-dimensional vector of unknown parameters expressed as $\theta = (\theta_1, \ldots, \theta_p)'$.

It is assumed that $lL(\theta|x)$ is regular with respect to all derivatives up to the third order. The joint cumulants of the derivatives of $lL(\theta|x)$ in Equations (4)-(7) are therefore defined as The second and third derivatives of the log-likelihood function are given by:

$$k_{ij} = \mathbb{E}\left[\frac{\partial^2 lL(\theta|x)}{\partial\theta_i\partial\theta_j}\right], \quad (4)$$

$$k_{ijl} = \mathbb{E}\left[\frac{\partial^3 lL(\theta|x)}{\partial\theta_i\partial\theta_j\partial\theta_l}\right], \quad (5)$$

$$k_{(ij,l)} = \mathbb{E}\left[\frac{\partial^2 lL(\theta|x)}{\partial\theta_i\partial\theta_j} \cdot \frac{\partial lL(\theta|x)}{\partial\theta_l}\right], \quad (6)$$

$$k_{ij}^{l} = \frac{\partial k_{ij}}{\partial\theta_l}, \quad (7)$$

where $i, j, l = 1, \ldots, p$. All four equations given by (4)-(7) are regarded as being of order $\mathcal{O}(n)$. Let $K = [-k_{ij}]$ represent the Fisher information matrix of θ for $i, j = 1, \ldots, p$. Cox and Snell (1968) demonstrated that when the sample data exhibit independence, though not necessarily an identical distribution, the bias of the r-th element of $\hat{\theta}_r$ in (8) can be written as:

$$\text{Bias}(\hat{\theta}_r) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} k^{ri}k^{jl}\left[0.5k_{ijl} + k_{(ij,l)}\right] + \mathcal{O}(n^{-2}), \quad (8)$$

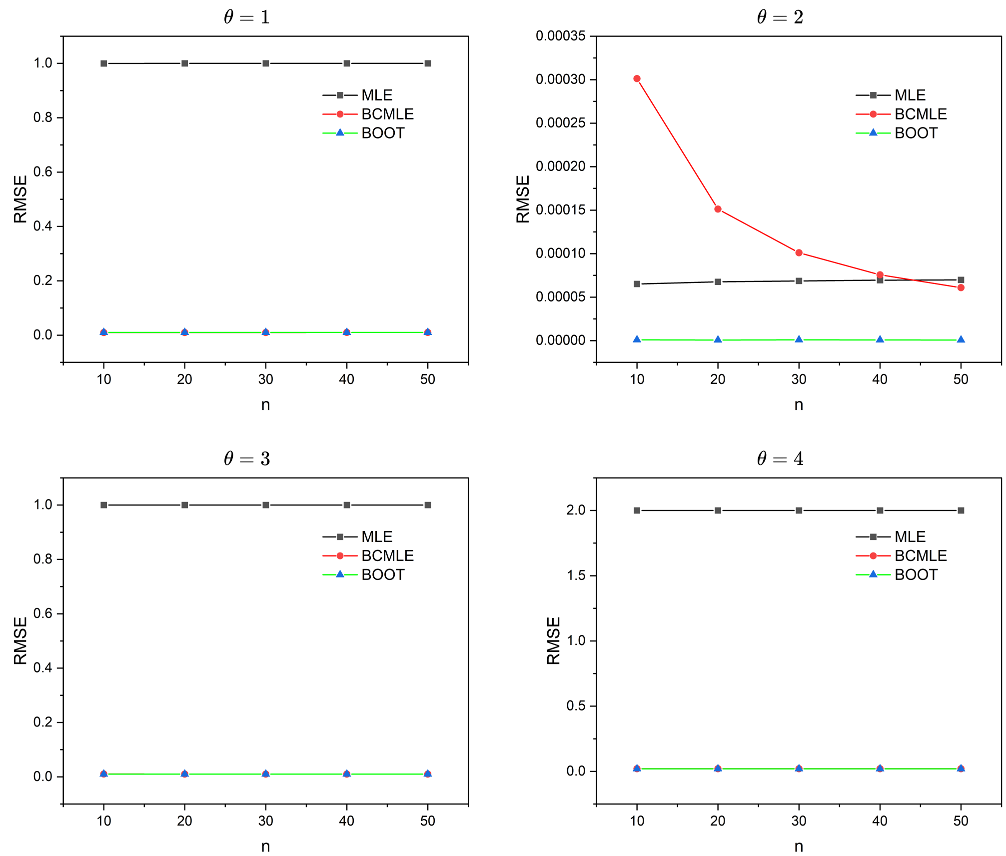

Figure 3. Plots Comparing the RMSEs of Three Different Estimation Methods for $\theta = 1, 2, 3, 4$

where $r = 1, \ldots, p$ and k^{ij} represents the (i, j)-th element of the reciprocal of the expected Fisher information matrix.

Cordeiro and Klein (1994) illustrated that when the sample data do not follow identical distributions, and all expressions in (4) through (7) are of order $\mathcal{O}(n)$, Equation (8) is subsequently expressed as:

$$\text{Bias}(\hat{\theta}_r) = \sum_{i=1}^{p} k^{ri} \sum_{j=1}^{p} \sum_{l=1}^{p} k^{jl} \left[k_{ij}^{(l)} - 0.5 k_{ijl} \right] + \mathcal{O}(n^{-2}),$$

for $r = 1, \ldots, p$. Determine a matrix $A^{(l)} = [a_{ij}^{(l)}]$ with its elements given by $a_{ij}^{(l)} = k_{ij}^{(l)} - 0.5 k_{ijl}$ for $i, j, l = 1, \ldots, p$. We have:

$$A = [A^{(1)} | A^{(2)} | \ldots | A^{(p)}], \quad \text{where} \quad A^{(l)} = [a_{ij}^{(l)}].$$

Hence, the bias for $\hat{\theta}$ is expressed in matrix form as:

$$\text{Bias}(\hat{\theta}) = K^{-1} A \cdot \text{vec}(K^{-1}) + \mathcal{O}(n^{-2}),$$

where the operator vec forms a column vector from a matrix by stacking the individual column vectors one below the other.

Therefore, the bias-corrected MLE of θ in (9), represented as $\hat{\theta}_{\text{BCMLE}}$, is given by:

$$\hat{\theta}_{\text{BCMLE}} = \hat{\theta} - \hat{K}^{-1} \hat{A} \cdot \text{vec}(\hat{K}^{-1}), \tag{9}$$

where $\hat{\theta}$ is the maximum likelihood estimator of θ, $\hat{K} = K|_{\theta=\hat{\theta}}$, and $\hat{A} = A|_{\theta=\hat{\theta}}$. It is important to observe that the bias of $\hat{\theta}_{\text{BCMLE}}$ follows a second-order pattern $\mathcal{O}(n^{-2})$.

For the bias-adjusted MLE of the ER distribution, we have $p = 1$, with $\theta = (\theta)'$. The joint cumulants of the derivatives of the log-likelihood function of θ ($lL(\theta|x)$) are obtained as follows:

$$k_{11} = -\frac{8n}{\theta^2} - \frac{6n(12\theta - \theta^4)}{(6 + \theta^3)^2},$$

$$k_{111} = \frac{16n}{\theta^3} - \frac{12n(\theta^6 - 42\theta^3 + 36)}{(6 + \theta^3)^3}.$$

In addition, we have:

$$K = \left[\frac{8n}{\theta^2} + \frac{6n(12\theta - \theta^4)}{(6 + \theta^3)^2} \right],$$

$$k_{11}^{(1)} = k_{111} = \frac{16n}{\theta^3} - \frac{12n(\theta^6 - 42\theta^3 + 36)}{(6+\theta^3)^3},$$

by using $a_{11}^{(1)} = k_{11}^{(1)} - 0.5k_{111}$.

$$A = [A^{(1)}], \quad \text{where} \quad A^{(1)} = [a_{11}^{(1)}].$$

$$A = 0.5\left[\frac{16n}{\theta^3} - \frac{12n(\theta^6 - 42\theta^3 + 36)}{(6+\theta^3)^3}\right].$$

Since the bias-corrected estimator, $\hat{\theta}_{\text{BCMLE}}$ in (9) lacks a closed-form solution, the nonlinear equation is solvable through the `optimize` function in the statistical software R. It should be noted that $\hat{\theta}_{\text{BCMLE}}$ is a bias-corrected MLE of θ to order $\mathbb{O}(n^{-1})$ and that its bias is of order $\mathbb{O}(n^{-2})$, because $\mathbb{E}[\hat{\theta}_{\text{BCMLE}}] = \theta + \mathbb{O}(n^{-2})$.

2.1.2 Parametric Bootstrap Method

Efron (1982) proposed a parametric bootstrap method as an alternative technique for generating bias-corrected estimators. This technique estimates the maximum likelihood of the dataset to generate pseudo-random samples from the distribution, referred to as parametric bootstrap samples. It uses these samples to estimate the bias and subsequently subtracts it from the MLE. The estimated bias of $\hat{\theta}_{\text{MLE}}$ is represented as:

$$\hat{B}(\hat{\theta}_{\text{MLE}}) = \frac{1}{B}\sum_{j=1}^{B}\left[\hat{\theta}_j - \hat{\theta}_{\text{MLE}}\right],$$

where $\hat{\theta}_j$ is the MLE of θ obtained from the j-th bootstrap sample, generated from (1) and using the $\hat{\theta}_{\text{MLE}}$ as the true value. Therefore, the parametric bootstrap bias-corrected estimator in (10) is given by:

$$\hat{\theta}_{\text{BOOT}} = 2\hat{\theta}_{\text{MLE}} - \frac{1}{B}\sum_{j=1}^{B}\hat{\theta}_j. \tag{10}$$

This technique presents an unbiased estimator with second-order bias correction.

3. RESULT AND DISCUSSION

3.1 Simulation Study

In this section, the aim of this research is to evaluate the performance of three estimators $\hat{\theta}$ ($\hat{\theta}_{\text{MLE}}, \hat{\theta}_{\text{BCMLE}}, \hat{\theta}_{\text{BOOT}}$) for the parameter θ of the ER distribution proposed in Section 3. A Monte Carlo simulation study was conducted using R version 4.3.3 to examine scenarios with small and moderate sample sizes ($n = 10, 20, 30, 40, 50$). The data were generated from the ER distribution using the acceptance-rejection method (sometimes called rejection sampling), and the parameter is fixed at $\theta = 1, 2, 3, 4$. For each combination of n and θ, the total number of Monte Carlo replications is set at $M = 10,000$, with $B = 1,000$ bootstrap replicates. The `optimize` function within the R program is utilized to derive the maximum likelihood estimate for the ER distribution.

To assess the accuracy of three estimators $\hat{\theta}$ ($\hat{\theta}_{\text{MLE}}, \hat{\theta}_{\text{BCMLE}}, \hat{\theta}_{\text{BOOT}}$) for the parameter θ of the ER distribution, we conducted an examination based on criteria such as average bias (AB) and root mean squared error (RMSE), which are calculated by

$$\text{AB}(\hat{\theta}) = \frac{1}{M}\sum_{i=1}^{M}(\hat{\theta}_i - \theta), \text{ and}$$

$$\text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\hat{\theta}_i - \theta)^2}, \text{ respectively.}$$

The results obtained from the Monte Carlo simulation, which investigate the average biases and root mean squared error (RMSE) of the estimates, are presented in Tables 1 and 2. Simulation results in Tables 1 and 2 indicate that the average biases of $\hat{\theta}_{\text{BCMLE}}$ and $\hat{\theta}_{\text{BOOT}}$ are smaller than the average biases of $\hat{\theta}_{\text{MLE}}$. The average biases of three estimators ($\hat{\theta}_{\text{MLE}}, \hat{\theta}_{\text{BCMLE}}, \hat{\theta}_{\text{BOOT}}$) appear negative for $\theta = 2, 3, 4$, except for $\theta = 1$. In most cases, the average biases of all estimators of θ decrease as the sample sizes increase. However, for $\theta = 1$, the average biases of all estimators of θ increase as the sample sizes increase.

Therefore, the estimators $\hat{\theta}_{\text{BCMLE}}$ and $\hat{\theta}_{\text{BOOT}}$ clearly outperform the estimator $\hat{\theta}_{\text{MLE}}$ in terms of bias. These estimators substantially reduce bias, particularly in small to moderate sample sizes. Accordingly, we examine these as more advantageous alternatives to $\hat{\theta}_{\text{MLE}}$ for estimating θ. This research observes that as the sample size increases, the adjusted estimates tend to be closer to the true parameter values in comparison to the unadjusted estimates. Furthermore, the corrected estimates provide root mean squared errors that are lower than those of the uncorrected estimates, indicating that the estimators $\hat{\theta}_{\text{BCMLE}}$ and $\hat{\theta}_{\text{BOOT}}$ result in a reduction in root mean squared error.

It is noteworthy that all estimators exhibit the property of consistency; specifically, the RMSE diminishes as the sample size increases. Therefore, the simulation results in Tables 1 and 2, corresponding to Figures 2 and 3, demonstrate the plots comparing the average bias and RMSEs of three different estimators: the maximum likelihood estimator (MLE), the bias-corrected MLE (BCMLE), and the parametric bootstrap bias-corrected estimator (BOOT). These results show a successful reduction of second-order bias, resulting in estimates that are closer to their true values.

3.2 Application

In this part, we demonstrate the practical application of three estimators ($\hat{\theta}_{MLE}, \hat{\theta}_{BCMLE}, \hat{\theta}_{BOOT}$) for the parameter θ of the ER distribution. The following real data set is considered. The dataset comprises the strength data of aircraft window glass reported by Fuller et al. (1994). The total sample size is 31.

Data: 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

The aforementioned data was analyzed by Alhyasat et al. (2021), and the results indicated that the dataset follows an ER distribution. The histogram showing the data fitted to the ER distribution is presented in Figure 4.

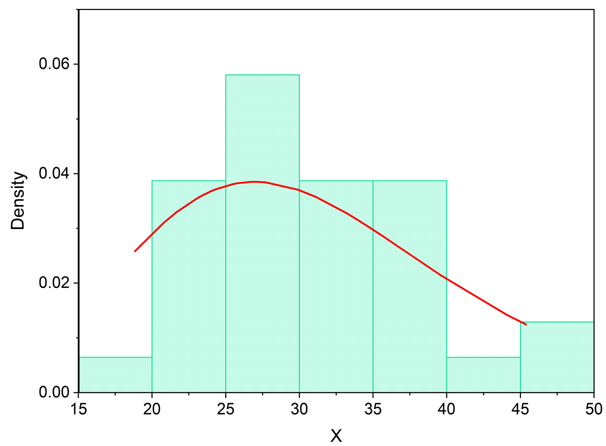

Figure 4. The Histogram of the Data Fits the ER Distribution.

Table 3 reported the point estimator of the ER distribution obtained using three methods: maximum likelihood method ($\hat{\theta}_{MLE}$), Cox-Snell method ($\hat{\theta}_{BCMLE}$), and parametric bootstrap method ($\hat{\theta}_{BOOT}$) for Fuller et al. (1994) data. The estimated parameters $\hat{\theta}_{BCMLE}$ and $\hat{\theta}_{BOOT}$ are smaller than $\hat{\theta}_{MLE}$, which suggests that the estimation by the maximum likelihood method overestimates θ. These results correspond with the simulation findings, which demonstrate that the estimators $\hat{\theta}_{BCMLE}$ and $\hat{\theta}_{BOOT}$ substantially reduce bias in instances of small or moderate sample sizes.

Table 3. Point Estimates for Fuller et al. (1994) Data

Estimator	θ
$\hat{\theta}_{MLE}$	0.2591
$\hat{\theta}_{BCMLE}$	0.2580
$\hat{\theta}_{BOOT}$	0.2581

4. CONCLUSIONS

In this study, we focus on estimating the parameter of the extended Rama distribution for small and moderate sample sizes using the maximum likelihood method. We introduce two bias-corrected methods, namely the Cox-Snell method and the parametric bootstrap method, to estimate the average bias and root mean square error associated with these estimators. Our analysis of numerical findings indicates that the recommended bias-corrected estimators outperform the maximum likelihood estimator in terms of bias and root mean square error, as observed in both simulation studies and real dataset analyses. Notably, as the sample size increases, the average bias and root mean square errors of all estimators tend to decrease. Furthermore, our investigation reveals that the Cox-Snell method is less effective than the parametric bootstrap method in reducing bias. However, it is observed that the average bias and root mean square error of the parametric bootstrap method exhibit fluctuations compared to those of the Cox-Snell method. Based on these findings, we recommend using the Cox-Snell method and the parametric bootstrap method as effective techniques to reduce biases associated with the maximum likelihood estimator, particularly for small and moderate sample sizes.

5. ACKNOWLEDGMENT

The author would like to thank the editor and the reviewers for their valuable comments and recommendations to improve this research. The author is grateful for the financial support from Faculty of Science, Silpakorn University, grant number SRIF-PRG-2566-10.

REFERENCES

Al-Shomrani, A. A. (2023). An Improvement in Maximum Likelihood Estimation of the Gompertz Distribution Parameters. *Journal of Statistical Theory and Applications*, **22**; 98–115

Alhyasat, K. M., K. Ibrahim, A. I. Al-Omari, and M. A. Abu Bakar (2021). Extended Rama Distribution: Properties and Applications. *Computer Systems Science and Engineering*, **39**(1); 55–67

Cordeiro, G. M. and R. Klein (1994). Bias Correction in ARMA Models. *Statistics & Probability Letters*, **19**(3); 169–176

Cox, D. R. and E. J. Snell (1968). A General Definition of Residuals. *Journal of the Royal Statistical Society. Series B (Methodological)*, **30**(2); 248–275

Deshpande, J. V. and S. G. Purohit (2005). *Lifetime Data: Statistical Models and Methods*. World Scientific, New Jersey

Efron, B. (1982). *The Jackknife, the Bootstrap and Other Resampling Plans*, volume 38. SIAM, Philadelphia, PA, USA

Fuller, E. J., S. W. Freiman, J. B. Quinn, G. D. Quinn, and W. C. Carter (1994). Fracture Mechanics Approach to the Design of Glass Aircraft Windows: A Case Study. *SPIE*, **2286**; 419–430

Lawless, J. F. (2002). *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons

Lehmann, E. L. (1999). *Elements of Large-Sample Theory*. Springer, New York

Lindley, D. V. (1958). Fiducial Distributions and Bayes' Theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, **20**(1); 102–107

Maghami, M. M. and M. Bahrami (2020). Improved Maximum Likelihood Estimation of Parameters in the Maxwell Distribution. *Journal of Statistical Modelling: Theory and Applications*, **1**(2); 99–113

Makalic, E. and D. F. Schmidt (2023). Maximum Likelihood Estimation of the Weibull Distribution with Reduced Bias. *Statistics and Computing*, **33**; 69

Mazucheli, J., S. R. Bapat, and A. F. B. Menezes (2020). A New One-Parameter Unit-Lindley Distribution. *Chilean Journal of Statistics*, **11**(1); 53–67

Mazucheli, J. and S. Dey (2018). Bias-Corrected Maximum Likelihood Estimation of the Parameters of the Generalized Half-Normal Distribution. *Journal of Statistical Computation and Simulation*, **88**(6); 1027–1038

Mazucheli, J., A. F. B. Menezes, F. Alqallaf, and M. E. Ghitany (2021). Bias-Corrected Maximum Likelihood Estimators of the Parameters of the Unit-Weibull Distribution. *Austrian Journal of Statistics*, **50**(3); 41–53

Mazucheli, J., A. F. B. Menezes, and S. Dey (2018). Improved Maximum-Likelihood Estimators for the Parameters of the Unit-Gamma Distribution. *Communications in Statistics-Theory and Methods*, **47**(15); 3767–3778

Mazucheli, J., A. F. B. Menezes, and S. Dey (2019). Bias-Corrected Maximum Likelihood Estimators of the Parameters of the Inverse Weibull Distribution. *Communications in Statistics-Simulation and Computation*, **48**(7); 2046–2055

Millar, R. B. (2011). *Maximum Likelihood Estimation and Inference with Examples in R, SAS and ADMB*. John Wiley & Sons

Saghir, A., A. Khadim, and Z. Lin (2017). The Maxwell Length-Biased Distribution: Properties and Estimation. *Journal of Statistical Theory and Practice*, **11**(1); 26–40

Schwartz, J. and D. E. Giles (2016). Bias-Reduced Maximum Likelihood Estimation of the Zero-Inflated Poisson Distribution. *Communications in Statistics-Theory and Methods*, **45**(2); 465–478

Shanker, R. (2017a). Rama Distribution and Its Application. *International Journal of Statistics and Applications*, **7**(1); 26–35

Shanker, R. (2017b). Rani Distribution and Its Application. *Biometrics & Biostatistics International Journal*, **6**(1); 256–265

Shanker, R., H. Fesshaye, and S. Selvaraj (2015). On Modeling of Lifetimes Data Using Exponential and Lindley Distributions. *Biometrics & Biostatistics International Journal*, **2**(5); 140–147

Shanker, R., K. K. Shukla, and H. Fesshaye (2017). A Generalization of Sujatha Distribution and Its Applications with Real Lifetime Data. *Journal of Institute of Science and Technology*, **22**(1); 66–83

Turhan, N. S. (2020). Karl Pearson's Chi-Square Tests. *Educational Research and Reviews*, **15**(9); 575–580

